Lie-type-like groups

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Abstract A finite simple group of Lie type in defining characteristic p has exactly two p-blocks, the principal block and a block of defect zero consisting of the Steinberg character whose degree is the p-part of the order of the group. In this paper we characterize finite groups G which have exactly the principal p-block and a p-block of defect zero consisting of an irreducible character of degree $|G|_p$.

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1 Introduction

Determining the number of p-blocks of a finite group is in general a very subtle task and extremly difficult to answer if at all. The easiest case, namely that G has only one block, has been settled completely by Harris in [10]. The next case, i.e., a complete characterization of groups G which have exactly two blocks does not seem to be accessible. In particular, the situation $\mathrm{O}_p(G) \neq 1$ seems out of reach. However if we restrict to the case that G has only the principal p-block and a p-block of defect zero (which implies $\mathrm{O}_p(G) = 1$) the methods are strong enough to determine the group up to some extent. This case is of particular interest since the canonical situation is that of a finite simple group of Lie type in defining characteristic p (see [12], section 8.5). In this case the block of defect zero consists of the Steinberg character which has degree $|G|_p$.

Definition 1.1 A finite group G is called Lie-type-like for the prime p if G has exactly the principal p-block and a p-block of defect zero with an irreducible ordinary character of degree $|G|_p$.

In this paper we prove the following two theorems where the second one depends on the classification of finite simple groups.

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Theorem 1.2 *Let G be a finite group with $\mathrm{F}^*(G) = \mathrm{F}(G)$ and let p be a prime dividing $|G|$. Then G is a Lie-type-like group for the prime p if and only if $G = HP$ where H is an elementary abelian normal r-subgroup ($r \neq p$ a prime) and P is a Sylow p-subgroup of G which acts regularly on the non-trivial elements of H.*

All groups occuring in Theorem 1.2 can be classified easily which leads to the following consequence.

Corollary 1.3 *Let G be a finite group with $\mathrm{F}^*(G) = \mathrm{F}(G)$ and let p be a prime dividing $|G|$. Then G is a Lie-type-like group for the prime p if and only if one of the following holds.*

(i) *$G = HP \leq GL(m, 2)$, where $H = \mathrm{F}_2^m$ and P is a Singer cycle in $GL(m, 2)$ of order a Mersenne prime p acting regularly on the non-trivial elements of H.*

(ii) *$G = HP$, where $H = \mathrm{F}_r$ with r a Fermat prime and $P = \mathrm{F}_r^*$ acts on H by multiplication.*

(iii) *$G = S_3$ or $(C_3 \times C_3).Q_8$, where Q_8 is the quaternion group of order 8 acting regularly on the non-trivial elements of $C_3 \times C_3$.*

Proof. The "if" part is clear, and it suffices to prove the "only if" part. By Theorem 1.2, we write $G = HP$ with $|P| = p^n$ and $|H| = r^m$. Since P acts regularly on the non-trivial elements of H we have $r^m - 1 = p^n$.

If $m = 1 = n$, then $r = 3$, $p = 2$ and $G = S_3$.

If $n = 1 < m$, then p is a Mersenne prime and P (of order p) is generated by a Singer cycle in $GL(m, 2)$ acting on $H = \mathrm{F}_2^m$.

If $m = 1 < n$, then $r = |P| + 1 = 2^n + 1$ is a Fermat prime and $P = \mathrm{F}_r^*$ acts on $H = \mathrm{F}_r$ by multiplication.

Thus we may assume that $m, n > 1$. In this case the only solution of $r^m - 1 = p^n$ is $3^2 - 1 = 2^3$ according to [19], and we get $G = (C_3 \times C_3).Q_8$ where Q_8 is the quaternion group of order 8. (Since p and r are primes the deep result of [19] can be avoided due to elementary calculations.) □

Theorem 1.4 *Let G be a finite group with $\mathrm{F}^*(G) \neq \mathrm{F}(G)$ and let p be a prime dividing $|G|$. Then G is a Lie-type-like group for the prime p if and only if one of the following holds.*

(i) *G is a finite simple group of Lie type in defining characteristic p.*

(ii) *$p = 3$ and $G = L_2(8).3$, the automorphism group of $L_2(8)$.*

(iii) *$p = 2$ and $G = S_6, M_{10}, A_8$ or $L_2(7)$.*

Note that $L_2(8).3 = {}^2G_2(3), S_6 = Sp(4, 2), A_8 = L_4(2)$ and $L_2(7) = L_3(2)$. Thus, as an immediate Corollary we have

Corollary 1.5 *If G is a non-solvable Lie-type-like group then G is a finite group of Lie type in defining characteristic p or $p = 2$ and $G = M_{10}$. Furthermore, G is simple for $p \geq 5$.*

Throughout this paper all groups are assumed to be finite and p always means a prime. By $\mathrm{Aut}(G)$ we denote the automorphism group and by $\mathrm{Bl}(G)$ the set of p-blocks of G. Sometimes we also use the notation $\mathrm{Bl}_p(G)$ to avoid ambiguity. We say that two blocks B_1 and B_2 are $\mathrm{Aut}(G)$-conjugate if there is a $\sigma \in \mathrm{Aut}(G)$ such that $B_1^\sigma = B_2$. Similar to the notation for conjugacy classes, we use $\mathrm{Cl}(\mathrm{Bl}(G))$ to denote the set of $\mathrm{Aut}(G)$-conjugate classes of p-blocks of G. Finally, $\mathrm{cd}(G)$ stands for the set of irreducible ordinary character degrees of G. For other notations, the reader is referred to the books [21], [22], [14].

2 Preliminaries

In this section we collect some general results on p-blocks of finite groups.

Lemma 2.1 *If $N \trianglelefteq G$ then $|\,\mathrm{Bl}(G)| \geq |\,\mathrm{Cl}(\mathrm{Bl}(N))|$. In particular, G has at least three p-blocks if $|\,\mathrm{Cl}(\mathrm{Bl}(N))| \geq 3$.*

Proof. This is obvious (see [21], Chap. 5, Lemma 5.3). □

Lemma 2.2 *We have $|\,\mathrm{Bl}(G)| \geq |\,\mathrm{Bl}(G/\,\mathrm{Z}(G))|$ with strict inequality if $\mathrm{O}_{p'}(\mathrm{Z}(G)) \neq 1$.*

Proof. Clearly, $|\,\mathrm{Bl}(G/\,\mathrm{O}_{p'}(\mathrm{Z}(G))| = |\,\mathrm{Bl}(G/\,\mathrm{Z}(G)|$. Moreover a p-block of $G/\,\mathrm{O}_{p'}(\mathrm{Z}(G))$ corresponds to a p-block of G by inflation. The second statement follows by the fact that G has a p-block whose characters do not restrict trivially on $\mathrm{O}_{p'}(\mathrm{Z}(G)) \neq 1$. □

Corollary 2.3 *Let $N \trianglelefteq G$ with $\mathrm{O}_{p'}(Z(N)) \neq 1$. If $|\,\mathrm{Bl}(N/\,\mathrm{Z}(N))| \geq 2$, then $|\,\mathrm{Bl}(G)| \geq 3$.*

Proof. According to Lemma 2.2 the normal subgroup N has at least three blocks which belong to different conjugacy classes. Thus the assertion follows by Lemma 2.1 □

Proposition 2.4 *Suppose that G has exactly two p-blocks.*

a) *If $\mathrm{F}^*(G) = \mathrm{F}(G)$ then $\mathrm{O}_{p'}(G) \neq 1$.*
b) *If $\mathrm{O}_{p'}(G) \neq 1$ then $\mathrm{O}_{p'}(G)$ is an abelian minimal normal subgroup of G.*
c) *If $\mathrm{F}^*(G) \neq \mathrm{F}(G)$ and p is odd then $\mathrm{O}_{p'}(G) = 1$.*

Proof. We put $H = \mathrm{O}_{p'}(G)$.
a) Obviously, $\mathrm{F}^*(G) = \mathrm{F}(G) = \mathrm{O}_p(G) \times N$ where $N \subseteq H$. If $H = 1$ then G has only one p-block by ([22], Corollary 9.21), since

$$C_G(\mathrm{O}_p(G)) = C_G(\mathrm{F}^*(G)) \subseteq \mathrm{F}^*(G) = \mathrm{O}_p(G),$$

a contradiction. Thus $H \neq 1$.

b) Note that the p-blocks of H which are covered by the same p-block of G are G-conjugate and that every p-block of H has only one irreducible character. Since G has exactly two p-blocks it follows that $|\mathrm{cd}(H)| \leq 2$. Thus, by Corollary 12.6 of [14], the group H is solvable.

We claim that $|\mathrm{cd}(H)| \neq 2$. If $|\mathrm{cd}(H)| = 2$ then the irreducible characters of H of same degree must be G-conjugate. It follows that H has only one irreducible character of degree 1. But this implies $|H : H'| = 1$, which contradicts the solvability of $H \neq 1$.

Thus we have $|\mathrm{cd}(H)| = 1$ and therefore H is abelian. Since G has exactly two p-blocks, all nontrivial irreducible characters of H must be G-conjugate. According to ([14], Corollary 6.33), we deduce that all nontrivial elements of H are G-conjugate. This means that H is an abelian minimal normal subgroup of G.

c) Assume $H \neq 1$. By part b), H is an elementary abelian r-subgroup for some prime $r \neq p$. Hence $H \leq \mathrm{F}(G) \leq \mathrm{F}^*(G)$. Let E(G) denote the layer of G.

If $N = H\mathrm{E(G)}$ then $H \leq Z(N)$ and $N/Z(N) \cong \mathrm{E(G)/Z(E(G))}$ (see [16], Section 6.5). Clearly, E(G)/Z(E(G)) is a direct product of some nonabelian simple groups. By [2], we have $|\operatorname{Bl}(N/Z(N))| = |\operatorname{Bl}(\mathrm{E(G)/Z(E(G))})| \geq 2$ and with Corollary 2.3 we obtain $|\operatorname{Bl}(G)| \geq 3$, a contradiction. Thus we have proved $H = 1$. □

Remark 2.5 The assumption that p is odd in part c) of Proposition 2.4 can not be removed. For example, take $N = M_{24} \times \mathbb{Z}_5$ and let $\mathbb{Z}_4$ act trivially on M_{24} and faithfully on $\mathbb{Z}_5$. Denote by G the corresponding semidirect product. Then it is easy to see that G has exactly two 2-blocks. However, $\mathrm{O}_{2'}(G) = \mathbb{Z}_5 \neq 1$.

Proposition 2.6 *Let G be a Lie-type-like group for the prime p. Then $\mathrm{F}^*(G)$ is either a nonabelian simple group or an abelian minimal normal subgroup of G.*

Proof. Let $L = \mathrm{F}^*(G)$. Since G has a p-block of defect zero, it follows that $\mathrm{O}_p(G) = 1$. By Proposition 2.4 b), we know that $\mathrm{O}_{p'}(G)$ is an abelian minimal normal subgroup of G. Therefore, if $L = \mathrm{F}(G)$ then $L = \mathrm{F}(G) = \mathrm{O}_{p'}(G)$ and we are done. Thus we may assume $L \neq \mathrm{F}(G)$ in the following.

We first suppose that p is odd. Since G is a Lie-type-like group, we have $|\operatorname{Cl}(\operatorname{Bl}(L))| = 2$ and that L has an irreducible character of degree $|L|_p$. According to Proposition 2.4 c) we have $\mathrm{O}_{p'}(G) = 1$. So the generalized Fitting subgroup L of G is a direct product of some non-abelian simple groups. Set $L = S_1 \times \cdots \times S_k$ with simple groups S_i. According to [2] we have $|\operatorname{Bl}(S_i)| \geq 2$ for $i = 1, \ldots, k$. Therefore $|\operatorname{Cl}(\operatorname{Bl}(L))| \geq 3$ if $k > 1$ and by Lemma 2.1, the group G has at least three p-blocks, a contradiction. Thus we have $k = 1$ which proves that L is a non-abelian simple group.

We now suppose $p = 2$. We claim $\mathrm{F}(G) = 1$. Suppose the contrary is true. Then we may assume that $|\operatorname{Bl}(L/Z(L))| = 1$, by Proposition 2.4 b) and Corollary 2.3. According to [10] the factor group $L/Z(L)$ is a direct product of copies of M_{22} and M_{24}. Now we choose $Z(L) \leq L_1 \leq L$, where L_1 is normal in L, such that $L_1/Z(L) \cong M_{22}$ or M_{24}. Observe that L and therefore L_1 must have an irreducible character χ of defect zero and degree $\chi(1) = |M_{22}|_2$ or $|M_{24}|_2$ respectively. But such a character does not exist since in the first case we have

$$L_1 = M_{22} \times Z(L) \qquad \text{or} \qquad L_1 = 3.M_{22} \times A, \quad A \text{ an elementary abelian 3-group,}$$

in the second

$$L_1 = M_{24} \times Z(L).$$

Thus the claim is proved.

It follows that L is a direct product of some non-abelian simple groups. Since neither M_{22} nor M_{24} has an irreducible character of 2-defect zero, we see that L has no composition factor isomorphic to M_{22} or M_{24}. Hence all composition factors of L have at least two 2-blocks. Using the same argument as in the p odd case, we get that L must be simple. □

Recall that X is said to be of type M if X is a quasisimple group with $X/\mathrm{Z}(X)$ isomorphic to a simple group M.

Lemma 2.7 *Let G be a finite group with exactly two 2-blocks. If $\mathrm{O}_{2'}(G) \neq 1$ then all components of $\mathrm{F}^*(G)$ are of type M_{22} or M_{24}.*

Proof. The statement of the Lemma is a direct consequence of Proposition 2.4 *b*), Corollary 2.3, and Theorem 1 of [10]. □

In the following we need the definition of a p-deficiency class which was introduced by R. Brauer in ([1], Sect. IV).

Definition 2.8 *If r is a non-negative integer then G is said to be of p-deficiency class r if all non-principal p-blocks of G have defect less than r.*

According to this definition, G is of p-deficiency class 0 if and only if G has only one p-block, and G is of p-deficiency class 1 if and only if all non-principal p-blocks of G are of defect zero. There are several equivalent conditions which characterize groups of p-deficiency class 1.

Lemma 2.9 ([10], Corollary 3.8). *A finite group G is of p-deficiency class 1 if and only if $C_G(x)$ is of p-deficiency class 0 for all $x \in G$ of order p.*

Using the same argument as in ([17], Proposition 2.5) we get

Lemma 2.10 *Let G be a finite group. Then the following two statements are equivalent.*

a) *G is of p-deficiency class 1.*
b) *$C_G(x)$ is of p-deficiency class 0 for every p-element $x \neq 1$ of G.*

Proof. According to Lemma 2.9 it suffices to show that a) implies b).

Let $1 \neq x$ be a p-element of G and let b be a p-block of $C_G(x)$ with defect group D. Clearly $x \in D$ and therefore $C_G(D) \leq C_G(x)$. Thus b is admissible (see [21], p. 322). By ([21], Chap. 5, Theorem 3.6 and Lemma 3.3), the block b^G is defined and $D \leq_G \delta(b^G)$, where $\delta(b^G)$ denotes a defect group of b^G. So the defect group of b^G is not trivial. Since

G is of p-deficiency class 1, it follows that b^G is the principal p-block of G. By Brauer's Third Main Theorem ([21], Chap. 5, Theorem 6.1), b is the principal p-block of $C_G(x)$. Thus $C_G(x)$ is of p-deficiency class 0. □

We would like to mention that the next result may be seen as a strengthening of a result of R. Brauer (see [2], Theorem 2.1).

Corollary 2.11 *Let G be a finite group and let $1 \neq x \in G$ be a p-element for an odd prime p. If $\mathrm{F}^*(C_G(x)) \neq \mathrm{F}(C_G(x))$ or $|\,\mathrm{Bl}(C_G(x))| \geq 2$ then G has a non-principal p-block which is not of p-defect zero.*

Proof. If $\mathrm{F}^*(C_G(x)) \neq \mathrm{F}(C_G(x))$ then the generalized Fitting group of $C_G(x)$ contains a non-trivial component N (i.e. a quasisimple subnormal subgroup). Since p is odd we have $|\,\mathrm{Bl}(N)| \geq 2$, by [2]. Hence $|\,\mathrm{Cl}(\mathrm{Bl}(N))| \geq 2$ as the principal block of N is stabilized by G. Thus $|\,\mathrm{Bl}(C_G(x))| \geq 2$ according to a repeated application of Lemma 2.1. By Lemma 2.10, we obtain that G is not of p-deficiency class 1 which proves the corollary. □

3 Proof of Theorem 1.2

Clearly, if G is a Frobenius group as in Theorem 1.2 then G has only the principal block and a block of defect zero whose irreducible character is of degree $|G|_p$ according to Clifford's theorem.

Thus we may assume that G is a finite Lie-type-like group and that $\mathrm{F}^*(G) = \mathrm{F}(G)$. We have to prove that G has the structure as given in Theorem 1.2. By Proposition 2.4 we know that $H = \mathrm{O}_{p'}(G)$ is an abelian minimal normal subgroup of G, hence is an r-group. Furthermore, since G has exactly two blocks, G acts transitively on the non-trivial elements of H, by ([22], Corollary 9.3). Let χ be any non-trivial irreducible character of H and let $T(\chi)$ denote the inertial group of χ.

Lemma 3.1 *$T(\chi)/H$ is a p'-group.*

Proof. Since H is a normal p'-subgroup, a block of G/H forms a block of G by inflation ([21], Chap. 5, Theorem 8.8). Thus G/H has exactly one block and $f_0 = \frac{1}{|H|}\sum_{h\in H} h$ is the block idempotent of the principal block of G. Note that the sum of block idempotents always equals to 1. Thus $f_1 = 1 - f_0$ is the block idempotent of the block of defect zero. Clearly, $1 \notin \mathrm{supp(f_1)}$ since otherwise G has two blocks of maximal defect. This implies that for each $1 \neq h \in H$ the centralizer $C_G(h)$ is a p'-group. Thus, by Brauer's permutation lemma ([21], Chap. 2, Lemma 2.19), the stabilizer $T(\chi)$ of χ is a p'-group for all non-trivial characters χ of H. □

Lemma 3.2 *We have $T(\chi) = H$.*

Proof. By the Fong-Reynolds theorem ([21], Chap. 5, Theorem 5.10), the blocks of $T(\chi)$ lying over χ are in one-to-one correspondence with the blocks of G lying over χ. Thus there is only one block of $T(\chi)$ lying over χ. Note that this block must be of defect zero, hence consists of an irreducible character, say ψ. This implies that

$$\psi|_H = e\chi \qquad \text{for some } e \in \mathbb{N}.$$

By ([14], Exercise 6.3 on page 95), we have $e^2 = |T(\chi) : H|$, and by Lemma 3.1, we know that $|T(\chi) : H|$ is a p'-number. On the other hand the non-principal block of G consists of an irreducible character of degree $|G|_p$ which implies that e is a power of p. Thus $T(\chi) = H$. □

To complete the proof of Theorem 1.2 note that $\Lambda = \chi^G$ is an irreducible character of G of p-defect zero and $|G : H| = \Lambda(1) = |G|_p$. Hence $G = HP$ where H is an elementary abelian r-group (r a prime) and P is a Sylow p-subgroup of G which acts regularly on $H \setminus \{1\}$. This completes the proof.

4 Blocks of non-abelian simple groups

In order to prove Theorem 1.4 we investigate finite non-abelian simple groups. According to the classification theorem of finite simple groups such a group is one of the following: an alternating group A_n ($n \geq 5$), a finite simple group of Lie type, or one of the 26 sporadic simple groups. Note that there are some isomorphic cases, such as $L_2(4) \cong L_2(5) \cong A_5$ and $L_2(9) \cong A_6$.

4.1 Alternating groups

Proposition 4.1 *Let $n \geq 5$ and let p be a prime dividing $|A_n|$. Then $|\operatorname{Cl}(\operatorname{Bl}(A_n))| \geq 3$ with the following exceptional cases*:

(i) $|\operatorname{Cl}(\operatorname{Bl}_p(A_n))| = 2$, *where* $p = 2, 3$ *and* $n = 5, 6, 7$;
(ii) $|\operatorname{Bl}_2(A_n)| = 2$ *for* $n = 5, 7, 8, 9, 11, 13$; *and*
(iii) $|\operatorname{Bl}_5(A_5)| = 2$.

Furthermore $G = A_5, A_6$ and A_8 are the only groups which have an irreducible character of degree $|G|_2$ in the cases (i) *and* (ii).

Proof. The complex irreducible characters of S_n are naturally labeled by the partitions of n. Let $\operatorname{Irr}(S_n) = \{[\lambda] \mid \lambda \vdash n\}$ where $\lambda \vdash n$ denotes a partition of n. Then the restriction $[\lambda]_{A_n}$ of $[\lambda] \in \operatorname{Irr}(S_n)$ to A_n is irreducible if λ is not self-conjugate. Using GAP [8], it is easy to check that the proposition is true for $n = 5, 6, 7$ and 8. So we may assume $n \geq 9$.

We first consider the case $n = p$. Let $\alpha = (p-2, 2)$ and $\beta = (p-3, 2, 1)$ be partitions of p. By the hook formula ([15], Theorem 20.1), we have $[\alpha](1) = \frac{p(p-3)}{2}$ and

$[\beta](1) = \frac{p(p-2)(p-4)}{3}$. Therefore,

$$\left\{\frac{p(p-3)}{2}, \frac{p(p-2)(p-4)}{3}\right\} \subseteq \mathrm{cd}(A_p).$$

Thus $A_n = A_p$ has at least two p-blocks of defect zero which are not $\mathrm{Aut}(G)$-conjugate and we are done in this case.

Similarly, the above statement holds for the cases $n = p+1$ and $n = p+2$. This follows from the partitions $\alpha' = (p, 1)$, $\beta' = (p-1, 1, 1)$, $\alpha'' = (p, 1, 1)$ and $\beta'' = (p-1, 2, 1)$, which imply that $\{p, \frac{p(p-1)}{2}\} \subseteq \mathrm{cd}(A_{p+1})$ and $\{\frac{p(p+1)}{2}, \frac{(p+2)p(p-2)}{3}\} \subseteq \mathrm{cd}(A_{p+2})$.

Next we assume that $n > p+2$. Let $x = (12 \cdots p)$. Then $C_{A_n}(x) = \langle x \rangle \times A_{n-p}$. By Theorem 1 of [10], A_{n-p} is not of p-deficiency class 0 for $n > 7$. So A_n is not of p-deficiency class 1, by Lemma 2.10. Since for all $n \geq 5$ the alternating group A_n always has a p-block of defect zero provided $p \geq 5$ (see [9], Corollary 1), we deduce that $|\mathrm{Cl}(\mathrm{Bl}(A_n))| \geq 3$ for $p \geq 5$.

We now consider the case $p = 3$ and $n = 3m + 1$ for some $m > 2$. Let B_1 and B_2 be the two 3-blocks of S_n corresponding to the 3-cores $(3, 1)$ and $(5, 3, 1^2)$, respectively. If $d(B_i)$ denotes the defect of B_i for $i = 1, 2$ then, by ([4], Proposition 2.12), we have $d(B_1) = \nu_3((n-4)!)$ and $d(B_2) = \nu_3((n-10)!)$, where ν_3 means the 3-adic valuation. Obviously, $d(B_1) > d(B_2)$.

Suppose that b_i is the 3-block of A_n covered by B_i for $i = 1, 2$. Since $|S_n : A_n| = 2$, it follows that $d(b_1) = d(B_1) > d(B_2) = d(b_2)$. Moreover, since

$$\nu_3(n!) > \nu_3((n-4)!) > \nu_3((n-10)!)$$

we see that neither b_1 nor b_2 is the principal p-block of A_n. Thus we have again $|\mathrm{Cl}(\mathrm{Bl}(A_{3m+1}))| \geq 3$.

Similarly, we obtain $|\mathrm{Cl}(\mathrm{Bl}(A_n))| \geq 3$ if $n = 3m-1$ or $n = 3m$ for some $m > 2$. In the first case we choose the 3-cores $(3, 1^2)$ and $(4, 2, 1^2)$, in the second the 3-cores $(4, 2)$ and $(3, 2^2, 1^2)$.

Finally, we assume that $p = 2$. GAP's library [8] shows that A_n has exactly two 2-blocks for $n = 9, 11, 13$. If n is even and $n \geq 10$, then we have at least three 2-cores for partitions of n, namely,

$$(\ ), (3, 2, 1), (4, 3, 2, 1).$$

Since $|S_n : A_n| = 2$, it follows that $|\mathrm{Cl}(\mathrm{Bl}(A_n))| \geq 3$ in this case. The same holds true if n is odd and $n \geq 15$. Indeed, we may choose the 2-cores $(1), (2, 1)$ and $(5, 4, 3, 2, 1)$. □

4.2 Simple groups of Lie type

For a finite group G we denote by $\pi(G)$ the set of primes dividing the order of G.

Proposition 4.2 *Let G be a finite simple group of Lie type in defining characteristic r and let p be a prime dividing $|G|$. If $p \neq r$ and $|\pi(G)| \leq 4$ then $|\mathrm{Cl}(\mathrm{Bl}(G))| \geq 3$, with the following exceptional cases*:

(i) $G \cong L_2(4) \cong L_2(5)$ *or* $L_2(8)$, $p = 3$, *and* $|\operatorname{Cl}(\operatorname{Bl}(G))| = 2$;

(ii) $G \cong L_2(4)$, $L_2(7)$ *or* $L_2(9)$, $p = 2$, *and* $|\operatorname{Cl}(\operatorname{Bl}(G))| = 2$. *Moreover, all of them have an irreducible character of degree* $|G|_2$.

Proof. By Theorem 2 of [27] or Theorem 1 of [13], either $G \cong L_2(q)$ or G is one of the following groups:

$$L_3(3), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(2), L_4(3),$$
$$U_3(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(2), U_4(3), U_5(2),$$
$$Sp_4(4), PSp_4(5), PSp_4(7), PSp_4(9),$$
$$O_7(2),\ O_8^+(2), {}^2F_4(2)', G_2(3), {}^3D_4(2),$$
$$Sz(8) \text{ and } Sz(32).$$

Step 1. Suppose that p is odd and $G \cong L_2(q)$. GAP's library shows that $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$ for $q \in \{4, 5, 7, 8, 9, 11\}$, with the exceptions $q = 4, 5, 8$ and $p = 3$. In the exceptional cases we have $|\operatorname{Cl}(\operatorname{Bl}(G))| = 2$. Thus we may assume $q = r^f > 11$. For the irreducible characters of $L_2(q)$ we refer the reader to ([7], Section 38).

Assume that q is odd and $4 \mid q-1$. According to the character table of $L_2(q)$ we have $\operatorname{cd}(G) = \{1, q, q+1, \frac{1}{2}(q+1), q-1\}$. If $p \mid q+1$ then G has two irreducible characters of p-defect zero and of distinct degrees. This implies $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$. To deal with the case $p \mid q-1$ note that $|\operatorname{Out}(G)| = (2, q-1)f$ (see [5], Table 5, Automorphisms and multipliers of the Chevalley groups). Since G has $\frac{q-1}{4} > 2f$ irreducible characters of degree $q-1$ we also have $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$ if $p \mid q-1$.

Next we assume that q is odd and $4 \nmid q-1$. Then $\operatorname{cd}(G) = \{1, q, q+1, q-1, \frac{1}{2}(q-1)\}$. Similarly as above, G has two irreducible characters of p-defect zero and distinct degrees if $p \mid q-1$ and G has $\frac{q-3}{4} > 2f$ irreducible characters of degree $q+1$. Both cases imply again $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$.

Finally, assume that $G = L_2(q)$, where $q = 2^f > 8$. Then $\operatorname{cd}(G) = \{1, q, q+1, q-1\}$. Moreover, G has $\frac{q-2}{2} > |\operatorname{Out}(G)| = f$ irreducible characters of degree $q+1$ and G has $\frac{q}{2} > |\operatorname{Out}(G)| = f$ irreducible characters of degree $q-1$. This proves $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$.

Step 2. Suppose that $p = 2$ and $G \cong L_2(q)$. Since $p \nmid q$ we have q odd. GAP's library [8] shows $|\operatorname{Cl}(\operatorname{Bl}(L_2(11)))| = 4$ and that the proposition holds for $q = 5, 7, 9$. For $q \geq 13$, we have $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$ by using the same argument as in Step 1.

Step 3. We now suppose that G is a group in the list of the beginning of the proof. By the Atlas [5] and the online data of [18] or using GAP [8], we deduce that $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$. In fact, for odd p, we have that G satisfies one of the following properties:

(1) G has at least three p-blocks of distinct defect; or

(2) G has at least two p-blocks of defect zero which are not $\operatorname{Aut}(G)$-conjugate.

For $p = 2$ the groups $L_3(3), U_3(3), U_4(3)$ and $G_2(3)$ are of 2-deficiency class 1. But it still holds that there are at least three $\operatorname{Aut}(G)$-conjugacy classes of 2-blocks. For the remaining groups, the situation is the same as for odd p. □

Proposition 4.3 *Let G be a finite simple group of Lie type in defining characteristic r and let p be a prime dividing $|G|$. If $p \neq r$ and $|\pi(G)| \geq 5$ then $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$.*

Proof. Let $\mathbf{G}$ be a simple simply connected algebraic group over the algebraic closure of a finite field $\mathbb{F}_q$, where q is a power of the prime r, and let $F : \mathbf{G} \to \mathbf{G}$ be the corresponding Frobenius endomorphism. Let $\widehat{G} = \mathbf{G}^F$ be the finite group of fixed points and assume that $\widehat{G}/Z(\widehat{G})$ is simple. Since all finite simple groups of Lie type apart from Tits' simple group can be constructed in such a way, we may assume $G = \widehat{G}/Z(\widehat{G})$. In the following we may exclude the Tits simple group since it contains at least three classes of p-blocks according to GAP. Let $\mathbf{T}$ be an F-stable maximal torus of $\mathbf{G}$ and let $(\mathbf{G}^*, F^*)$ be the dual pair of $(\mathbf{G}, F)$ with respect to $\mathbf{T}$ (see [6], Definition 13.10). Similarly, denote $\widehat{G^*} = \mathbf{G}^{*F}$ and note that $|G| = |(\widehat{G^*})'|$.

Recall that a Lusztig series $\mathcal{E}(\widehat{G}, s)$ associated to the geometric conjugacy class (s) of a semisimple element $s \in \widehat{G^*}$ is the set of irreducible characters of $\widehat{G}$ which occur in some Deligne-Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(\theta)$, where $\theta \in \mathrm{Irr}(T)$ and $(\mathbf{T}, \theta)$ is of the geometric conjugacy class associated to (s) (see [6], Definition 13.16). Lusztig's fundamental result asserts that Lusztig series associated to various geometric conjugacy classes of semisimple elements of $\widehat{G^*}$ form a partition of $\mathrm{Irr}(\widehat{G})$ (see [6], Proposition 13.17). Let $\sigma \in \mathrm{Aut}(\widehat{G})$. Then σ extends to a bijection morphism $\sigma_1 : \mathbf{G} \to \mathbf{G}$ of algebraic groups which commutes with F (see also the remark after Corollary 2.5 of [23]). By ([23], Corollary 2.4), σ_1 is compatible with Lusztig series and preserves the orders of the semisimple elements labelling them.

Let s be a semisimple p'-element of $\widehat{G^*}$. By ([6], Theorem 13.23), there is a bijection between $\mathcal{E}(\widehat{G}, s)$ and the set of unipotent characters of $C = C_{\mathbf{G}^*}(s)^{F^*}$. Therefore, to every conjugacy class (s) corresponds a so-called semisimple irreducible character $\chi_s \in \mathcal{E}(\widehat{G}, s)$ which is in correspondence with the trivial character of C. Furthermore, if $1 \neq s$ is contained in the derived subgroup $(\widehat{G^*})'$ then $Z(\widehat{G}) \subseteq \ker(\chi_s)$ by ([24], Lemma 4.4). So χ_s can be viewed as an irreducible character of G. Since $|\pi(G)| \geq 5$, we can take at least three $\{p, r\}'$-elements of $(\widehat{G^*})'$ of different prime orders, say s_i for $i = 1, 2, 3$. Thus the semisimple characters χ_{s_i} corresponding to s_i are actually irreducible characters of G and of $\hat{G}$. As characters of G they are in blocks, say b_i of G. In addition, these characters remain irreducible modulo p by ([11], Proposition 1). Since $G = \widehat{G}/Z(\widehat{G})$ it follows that each b_i is contained in a (unique) block B_i of $\hat{G}$ (see [21], Chap. 5, Theorems 8.8 and 8.10).

Furthermore, the set

$$\mathcal{E}_p(\widehat{G}, s) := \bigcup_{t \in C_{\mathbf{G}^*}(s)_p^{F^*}} \mathcal{E}(\widehat{G}, st)$$

is a union of p-blocks of $\widehat{G}$ according to a fundamental result ([3], Théorème 2.2) due to Broué and Michel. With the observation above, the $\mathcal{E}(\widehat{G}, s_i)$ and hence the blocks B_i of $\widehat{G}$ containing the χ_{s_i} lie in different orbits under the action of the automorphism group $\mathrm{Aut}(\widehat{G})$. Since $\mathrm{Aut}(G)$ can be viewed as a subgroup of $\mathrm{Aut}(\widehat{G})$, we conclude that the blocks b_i of G containing the χ_{s_i} lie in different orbits under the action of $\mathrm{Aut}(G)$. Thus $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$. □

4.3 Sporadic simple groups

Proposition 4.4 *Let G be one of the* 26 *sporadic simple groups. Then* $|\operatorname{Cl}(\operatorname{Bl}(G))| \geq 3$ *except the following cases*:

(i) $G = M_{11}$ *and* Co_3 *with two* 3*-blocks*;

(ii) $G = M_{12}, J_2, Co_1, Co_2, HS, Ru, Suz$ *and* B *with two* 2*-blocks, none of which is Lie-type-like for the prime* 2;

(iii) $G = M_{22}$ *or* M_{24} *with only one* 2*-block.*

Proof. This is checked by using GAP [8]. □

5 Proof of Theorem 1.4

In order to prove Theorem 1.4 we put $L = F^*(G)$ and suppose that G is a Lie-type-like group for the prime p. By Proposition 2.6, the group L is simple. We first assume that p is odd. According to the results of section 4 we know that L is either a simple group of Lie type in defining characteristic p or one of the groups $A_7, L_2(5), L_2(8), M_{11}, Co_3$. Moreover in the last five cases we have $p = 3$.

Since $C_G(L) \leq L$ we obtain $G \leq \operatorname{Aut}(L)$. The case $L = A_5$ can be ruled out since both A_5 and $\operatorname{Aut}(A_5)$ have three 3-blocks. Furthermore, neither A_7, S_7 nor Co_3 has a 3-block of defect zero. The group M_{11} does not have an irreducible character of degree $|M_{11}|_3$. So it remains to deal with the case $L = L_2(8)$. However, if this is the case, the group G has to be the automorphism group of $L_2(8)$ and we have proved part (ii) of the theorem.

Now suppose that L is a simple group of Lie type in defining characteristic p. Let St be the Steinberg character of L. By [25] and [26] the character St extends to G. Since G is a Lie-type-like group for the prime p, it follows that G has a block of defect zero which must cover the p-block of L containing St. Now Clifford's theorem forces $|G/L| = 1$, i.e., $G = L$, and we have part (i) of the theorem.

Finally it remains to consider the case $p = 2$ and we may assume that L is not a simple group of Lie type in characteristic 2. By the results in section 4, L is one of the groups A_6, A_8 or $L_2(7)$. According to the Atlas [5] we see that G is S_6, M_{10}, A_8 or $L_2(7)$ and we have part (iii) of the theorem.

To finish the proof we consider the other direction of equivalence. If G is one of the cases in (ii) or (iii) then GAP's library shows that G is a Lie-type-like group. As already mentioned in the introduction simple groups of Lie type in defining characteristic p are Lie-type-like groups for p. This accounts for (i) and finishes the proof.

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