A GENERALIZATION OF MURAI'S CONJECTURE

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Abstract We generalize Murai's conjecture on an upper bound for the number of irreducible p-Brauer characters in the principal block to an arbitrary block. We prove that the new conjecture has an affirmative answer for tame blocks and blocks with cyclic defect groups. In addition we confirm Murai's conjecture for symmetric and alternating groups.

Keywords: finite group, irreducible Brauer character, block, Cartan matrix

MSC2010: 20C20

1. Introduction

Throughout the paper p is always a prime and G a finite group. Let $|G|_{p'}$ denote the p'-part of $|G|$ and

$$G_{p'} = \{g \mid g \in G,\, g \text{ is a } p'\text{-element}\}$$

The first and second author were supported by NSFC (12171211) and the Natural Science Foundation of Jiangxi Province (20192ACB21008), and the third author was supported by NSFC (12201155).

the set of p-regular elements in G. By $\mathrm{IBr}_p(G)$ and $\mathrm{IBr}_p(B)$ we denote the set of irreducible p-Brauer characters of G, resp. of a p-block B of G with respect to a sufficiently large field K of characteristic p. Moreover, by C_B we always denote the Cartan matrix of a p-block B.

Let $l(B) = |\,\mathrm{IBr}_p(B)|$, $k(B) = |\,\mathrm{Irr}_{\mathbb{C}}(B)|$ and let B_0 be the principal p-block of G.

In [14], Murai conjectured that always $\quad l(B_0) \leq \frac{|G_{p'}|}{|G|_{p'}}$.

As Maurai carried out in his paper, an affirmative answer has many interesting consequences. For instance, Brauer's conjecture $k(B) \leq |D|$, where D is the defect group of B, holds for principal p-blocks ([14], Proposition 1.2). In particular, Brauer's conjecture holds true for any p-block of a p-solvable group ([14], Proposition 1.3).

To be brief we put $m(G) = m_p(G) = \frac{|G_{p'}|}{|G|_{p'}}$. Note that $p \nmid m_p(G)$, by ([6], Lemma 15.14).

Proposition 1.1. *If $P \in \mathrm{Syl}_p(G)$, then*

$$m(G) \equiv m(N_G(P)) \not\equiv 0 \bmod p.$$

Proof. We may assume that $N = N_G(P) < G$ and proceed by induction on the order $|G|$ of G. Suppose that $Z \leq Z(G)$ is a p-group. By ([14], Lemma 2.1), we have $m(G) \geq m(G/Z)$. On the other hand, a direct calculation shows that $m(G/Z) \geq m(G)$. Hence $m(G/Z) = m(G)$. So we may assume that $Z(G)$ is a p'-group.

Let $\{x_i \mid i \in I\} \subseteq P$ be a complete set of representatives of the conjugacy classes in G consisting of p-elements. Then, by ([14], Formula (1.2.2)), we have

$$\begin{aligned} 0 \equiv |G|_p &= \textstyle\sum_i \frac{|G|_p}{|C_G(x_i)|_p} m\left(C_G\left(x_i\right)\right) \\ &\equiv m(G) + \textstyle\sum_{1 \neq x_i \in Z(P)} m\left(C_G\left(x_i\right)\right) \quad \bmod p. \end{aligned}$$

Similarly, let $\{y_j \mid j \in J\} \subset P$ be a complete set of representatives of the conjugacy classes in N consisting of p-elements. Then

$$0 \equiv |N|_p \equiv m(N) + \textstyle\sum_{1 \neq y_j \in Z(P)} m\left(C_N\left(y_j\right)\right) \bmod p.$$

An application of Burnside's Lemma ([19], Lemma 10.20) shows that $Z(N)$ is a p'-group and that those x_i and y_j in $Z(P)$ can actually be chosen to be the same. Thus

$$m(G) + \sum_{1 \neq x_i \in Z(P)} m\left(C_G\left(x_i\right)\right) \equiv m(N) + \sum_{1 \neq x_i \in Z(P)} m\left(C_N\left(x_i\right)\right) \bmod p.$$

Note that

$$P \le C_N(x_i) = N \cap C_G(x_i) = N_{C_G(x_i)}(P).$$

Hence, by induction, we get

$$m(C_N(x_i)) = m(N_{C_G(x_i)}(P)) \equiv m(C_G(x_i)) \bmod p,$$

from which the assertion follows.

□

2. A generalization of Murai's conjecture.

Let B be a p-block of G. For $\{\beta_1, \beta_2, \ldots, \beta_l\} = \mathrm{IBr}_p(B)$ we put

$$\gamma_{ij} = \langle \beta_i, \beta_j \rangle^\circ = \frac{1}{|G|} \sum_{x \in G_{p'}} \beta_i(x)\beta_j(x^{-1}).$$

Note that $\Gamma_B = (\gamma_{ij})$ is the inverse of the Cartan matrix C_B of B ([4], Chap. IV, Lemma 3.7). If $B = B_0$ is the principal block, then $\beta_1 = 1_G$ will always denote the trivial Brauer character.

Lemma 2.1. *We have* $\gamma_{11}|G|_p = m(G)$.

Proof. This follows immediately by

$$\gamma_{11} = \langle 1_G, 1_G \rangle^\circ = \frac{1}{|G|} \sum_{x \in G_{p'}} 1_G(x) = \frac{|G_{p'}|}{|G|}.$$

□

Example 2.2. Now let $G = \mathrm{SL}(2,5)$ and $p = 2$. Then the principal 2-block B_0 of G has 3 irreducible Brauer characters, and

$$C_{B_0} = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 4 & 2 \\ 4 & 2 & 4 \end{pmatrix}$$

(see for instance ([6], Example 13.9)). For its inverse one easily computes

$$C_{B_0}^{-1} = \begin{pmatrix} \frac{3}{8} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}.$$

Thus, by Lemma 2.1, we have

$$l(B_0) = 3 = \frac{3}{8} \cdot 8 = \gamma_{11}|G|_2 = m_2(G) = m(G).$$

Let $p^{a(\beta)}$ denote the Hilbert divisor of $\beta \in \mathrm{IBr}_p(G)$ (for the definition and facts on Hilbert divisors we refer to [11]). If $\mathrm{IBr}_2(G) = \{1_G = \beta_1, \beta_2, \beta_3\}$, then $a(\beta_1) = 3$ and $a(\beta_i) = 2$ for $i = 2, 3$. Thus

$$\gamma_{ii} 2^{a(\beta_i)} = \frac{1}{2} \cdot 4 = 2 < l(B_0)$$

for $i = 2, 3$. However, in general $\gamma_{\beta\beta} \cdot p^{a(\beta)} < l(B)$ for $a(\beta) < d$, where d is the defect of B, does not always hold true.

Note that, by the proof of ([11], Theorem 2.1 a)), we always have $p^{a(\beta)}\gamma_{\beta\beta} \in \mathbb{N}$ for $\beta \in \mathrm{IBr}_p(G)$. Based on many examples we conjecture the following.

Conjecture 2.3. Let B be a p-block of defect d. Then

$$l(B) \leq p^d \gamma_{\beta\beta}$$

for all $\beta \in \mathrm{IBr}_p(B)$.

Conjecture 2.3 means that if B_0 is the principal p-block, then

$$l(B_0) \leq |G|_p \gamma_{11} = m_p(G)$$

by Lemma 2.1. So this is Murai's conjecture. Furthermore, by Example 2.2, we have $|G|_2\gamma_{22} = |G|_2\gamma_{33} = 2^3 \cdot \frac{1}{2} = 4 > l(B_0) = 3$.

Question 2.4. We may ask here the question: Is there always a $\beta \in \mathrm{IBr}_p(B)$ with $\gamma_{\beta\beta} \leq 1$?

Suppose that $\beta \in \mathrm{IBr}_p(B)$ is liftable to $\chi \in \mathrm{Irr}(B)$. Then

$$1 = \langle \chi, \chi \rangle = \gamma_{\beta,\beta} + \frac{1}{|G|} \sum_{g \ p\text{-singular}} \chi(g)\overline{\chi(g)}.$$

Since both parts are real and non-negative, we get $\gamma_{\beta,\beta} \leq 1$. Thus, in this case (in particular, if G is p-solvable or if B is principal), Conjecture 2.3 implies $l(B) \leq |D|$.

Remark 2.5. In general, the smallest value $p^d\gamma_{\beta\beta}$ is not always reached by a height zero character β. As an example the non-principal 2-block of A_9 of defect 3 may serve. It has 3 irreducible Brauer characters, say β_i of degree $8, 48$ and 160 and of height $0, 1$ resp. 2. The corresponding Hilbert divisors are $8, 4, 2$. One easily computes that $2^3\gamma_{\beta_i\beta_i} = 5, 4, 8$.

We would like to mention here that Malle and Robinson conjectured in [12] the upper bound

$$l(B) \leq p^{s(B)},$$

where $s(B)$ denotes the sectional p-rank of a defect group of B.

For lower bound of $l(B)$, in [8] Holm and the second author asked the question whether $l(B) \geq \frac{\operatorname{tr} C_B}{p^d}$ always holds true, where tr stands for the trace. In [16] Navarro and Sambale presented as counterexamples the principal 2-block of $Sz(32).5$ and $\mathrm{PSp}_4(4).4$. However, for p-solvable groups, we have indeed $l(B) \geq \frac{\operatorname{tr} C_B}{p^d}$, since $c_{\beta\beta} \leq p^d$ for $\beta \in \mathrm{IBr}_p(B)$, by [7].

Proposition 2.6. *Let G be a p-solvable group and let B be a p-block of G with defect d. Then $\operatorname{tr} C_B = l(B)p^d$ if and only if $l(B) = 1$.*

Proof. The assertion is clear if $l(B) = 1$. Now suppose that $l(B) > 1$ and

$$\sum_{\beta \in \mathrm{IBr}_p(B)} c_{\beta\beta} = l(B)p^d.$$

This forces $c_{\beta\beta} = p^d$ for all β, since $c_{\beta\beta} \leq p^d$. By ([11], Lemma 2.8), there exists $\beta \in \mathrm{IBr}_p(B)$ such that $c_{\beta\beta} < p^{a(\beta)} \leq p^d$, a contradiction. □

For the reader's convenience we recall a result on positive definite symmetric matrices which seems to be well known.

Lemma 2.7. *Let $A = (a_{ij})_{1 \leq i,j \leq l}$ be a positive definite symmetric matrix over the real numbers of type (l, l). Then*

$$\det A \leq \prod_{i=1}^{l} a_{ii}.$$

Proof. We may assume that $l \geq 2$. Let

$$A = \begin{pmatrix} A_1 & v \\ v^t & a_{ll} \end{pmatrix}$$

where $v = (a_{1l}, a_{2l}, \dots, a_{(l-1)l})^t$ and A_1 is of type $(l-1, l-1)$. Since A is positive definite, A_1 as a principal minor of A is positive definite as well. In particular, $\det A_1 > 0$ and A_1 is invertible. Hence

$$\begin{aligned} \det A &= \det A_1 \cdot \det \begin{pmatrix} E & A_1^{-1}v \\ v^t & a_{ll} \end{pmatrix} \\ &= \det A_1 \cdot \det \begin{pmatrix} E & A_1^{-1}v \\ 0 & a_{ll} - v^t A_1^{-1} v \end{pmatrix} \\ &= (\det A_1)(a_{ll} - v^t A_1^{-1} v). \end{aligned}$$

Now $v^t A_1^{-1} v \geq 0$, since A_1 is positive definite. Thus $\det A \leq (\det A_1) a_{ll}$ and by an inductive argument we obtain the assertion. □

Corollary 2.8. *Let B be a p-block of G with Cartan matrix $C_B = (c_{\alpha\beta})$, where $\alpha, \beta \in \mathrm{IBr}_p(B)$. Then* $\det C_B \leq \prod_{\beta \in \mathrm{IBr}_p(B)} c_{\beta\beta}$.

Proof. Since C_B is positive definite ([10], Lemma 2.3), we may apply Lemma 2.7. □

Theorem 2.9. *Let B be a p-block of defect d. Then* $\operatorname{tr} C_B^{-1} \geq \frac{l(B)}{p^d}$ *with equality if and only if $l(B) = 1$.*

Proof. Let $C = C_B$. The first statement follows by

$$p^d \operatorname{tr} C^{-1} = \sum_{\beta \in \mathrm{IBr}_p(B)} p^d \gamma_{\beta\beta} \geq \sum_{\beta \in \mathrm{IBr}_p(B)} 1 = l(B),$$

since $p^d \gamma_{\beta\beta} \in \mathbb{N}$.

Clearly, if $l(B) = 1$, then $\operatorname{tr} C^{-1} = \frac{l(B)}{p^d}$. For the converse, we write $l = l(B)$ and denote by $p^{d_1}, \ldots, p^{d_l}$ the elementary divisors of C, where $d_1 \leq \cdots \leq d_{l-1} < d_l$ (for the last inequality, see ([4], Chap. IV, Theorem 4.16)). Thus $\det(C) = p^{d_1} \cdots p^{d_l}$. As already mentioned, we furthermore have $p^d \gamma_{\beta\beta} \in \mathbb{N}$.

Suppose that $\operatorname{tr} C^{-1} = \frac{l(B)}{p^d}$. Thus $\gamma_{\beta\beta} = \frac{1}{p^d}$ for all $1 \leq i \leq l$. Note that C^{-1} is also positive definite. Thus, by Lemma 2.7, we get $\det(C^{-1}) \leq (\frac{1}{p^d})^l$. However, this is not possible unless $l = 1$, since $\det(C^{-1}) = \prod_{i=1}^{l} \frac{1}{p^{d_i}}$. This finishes the proof. □

Observe that an affirmative answer of Conjecture 2.3 will provide a new lower bound for $\operatorname{tr} C_B^{-1}$.

Remark 2.10. If Conjecture 2.3 holds true, then $\operatorname{tr} C_B^{-1} \geq \frac{l(B)^2}{p^d}$, since $p^d \gamma_{\beta\beta} \in \mathbb{N}$ for all $\beta \in \mathrm{IBr}_p(B)$.

Clearly, $\operatorname{tr} C_B^{-1} \geq \rho(C_B^{-1})$ where $\rho(C_B^{-1})$ denotes the Frobenius eigenvalue of C_B^{-1}. Thus we may ask whether

$$\rho(C_B^{-1}) \geq \frac{l(B)}{p^d}$$

which is equivalent to

$$\mu(C_B) l(B) \leq p^d,$$

where $\mu(C_B)$ is the smallest eigenvalue of C_B. Note that there are examples in which $\rho(C_B) \not\leq l(B) p^d$ as shown in [16].

Examples 2.11. a) Let $p = 2$ and $G = \mathrm{PSL}(2,8)$ so that the inverse of the Cartan matrix of the principal 2-block is

$$\begin{pmatrix} 7/8 & -1/4 & -1/4 & -1/4 & -1/2 & -1/2 & -1/2 \\ -1/4 & 3/2 & -1/2 & -1/2 & 1 & -1 & 0 \\ -1/4 & -1/2 & 3/2 & -1/2 & 0 & 1 & -1 \\ -1/4 & -1/2 & -1/2 & 3/2 & -1 & 0 & 1 \\ -1/2 & 1 & 0 & -1 & 2 & 0 & 0 \\ -1/2 & -1 & 1 & 0 & 0 & 2 & 0 \\ -1/2 & 0 & -1 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

Conjecture 2.3 leads to $l(B_0) \leq 7$ since $\gamma_{ii} = \frac{7}{8}, 3/2$ or 2. According to the Malle-Robinson conjecture we only get $l(B_0) \leq 8$. Actually $l(B_0) = 7$.

b) Let $G = A_4$ and $p = 2$. For the principal 2-block we have

$$l(B_0) = 3 \text{ and } p^d\gamma_{ii} = 4 \cdot \frac{3}{4} = 3$$

for all i. Thus the bound in Conjecture 2.3 is reached for all i. Note that the Malle-Robinson conjecture only leads to $l(B_0) \leq 4$.

c) Let B be a p-block of defect $d \geq 2$ with cyclic defect group and suppose that the Brauer tree is a star with exceptional vertex in the center. Let $e = l(B) \geq 2$ and $m = \frac{p^d-1}{e}$. For all i we have in this case

$$\gamma_{ii}p^d = (e-1)m + 1 = p^d - m = \frac{e-1}{e}p^d + \frac{1}{e} > \frac{e-1}{e}p^d \geq \frac{p^d}{2} \geq p,$$

since $e, d \geq 2$. Note that the sectional p-rank of a cyclic p-group is one. Thus the Malle-Robinson conjecture is stronger than our Conjecture 2.3.

3. Relations between the Cartan matrix and its inverse

Recall that the *Schur product of matrices*, denoted by $*$, is defined as the componentwise multiplication, i.e., if $A = (a_{ij})$ and $B = (b_{ij})$, then $A * B = (a_{ij}b_{ij})$. Now let $C_B = (c_{\alpha\beta})_{\alpha,\beta \in \mathrm{IBr}_p(B)}$ be the Cartan matrix of a p-block B with $l = l(B)$. To be brief we put $C = C_B$ in this section. Since C and C^{-1} are positive definite, we get that $C * C^{-1}$ is positive definite as well by the Schur product theorem ([9], Theorem 5.2.1). If I_l denotes the identity matrix of degree l, then we have the following.

Theorem 3.1. *$C * C^{-1} - I_l$ is positive semidefinite; i.e., $C * C^{-1} \succeq I_l$ in the positive semidefinite partial order.*

Proof. By ([9], Theorem 5.4.3), the smallest eigenvalue of $C * C^{-1}$ is 1. Since $C * C^{-1}$ is positive definite, the assertion follows. $\square$

Corollary 3.2. *For any $\beta \in \mathrm{IBr}_p(B)$ we have $c_{\beta\beta}\gamma_{\beta\beta} \geq 1$ with equality if and only if $l(B) = 1$. In particular,* $\mathrm{tr}\,(C * C^{-1}) \geq l(B)$.

Proof. Let $x = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at position β. By Theorem 3.1, we get

$$c_{\beta\beta}\gamma_{\beta\beta} = x(C * C^{-1})x^t \geq xI_l x^t = \langle x, x\rangle = 1.$$

Suppose that $c_{\beta\beta}\gamma_{\beta\beta} = 1$. Since $p^{a(\beta)}\gamma_{\beta\beta} \in \mathbb{N}$, we get $p^{a(\beta)} = nc_{\beta\beta}$ for some $n \in \mathbb{N}$.

Clearly, if $l(B) = 1$, then $c_{\beta\beta}\gamma_{\beta\beta} = 1$. To see the converse, suppose that $l := l(B) \geq 2$. In the following we use C and C_1 in Lemma 2.7 instead of A and A_1. Since $\gamma_{ll} = \frac{\det C_1}{\det C}$ and $\det C = \det C_1(c_{ll} - v^t C_1^{-1} v)$, we have

$$c_{ll}\gamma_{ll} = c_{ll} \cdot \frac{\det C_1}{\det C_1\left(c_{ll} - v^t C_1^{-1} v\right)} = \frac{c_{ll}}{c_{ll} - v^t C_1^{-1} v} > 1.$$

(Note that in the proof of Lemma 2.7, we have $v \neq 0$ by the indecomposability of C which follows from the fact that B is a p-block of G. Since C_1 is positive definite, $v^t C_1 v > 0$.) □

Clearly, if $m(G) \geq \mathrm{tr}\,(C * C^{-1})$ for the principal block of G, then Murai's conjecture holds true for G. Unfortunately, there are examples, even with a cyclic defect group, with $m(G) \leq \mathrm{tr}\,(C * C^{-1})$. As an example the group S_4 for $p = 3$ may serve. Actually, $m_3(S_4) = 2$ and $\mathrm{tr}(C * C^{-1}) = \frac{8}{3}$.

Corollary 3.3. *If $1 = 1_G$ is the trivial character, then*

$$c_{11} \geq \frac{|G|}{|G_{p'}|} = \frac{|G|_p}{m(G)}$$

with equality if and only if G is p-nilpotent.

Proof. By Corollary 3.2, we have $c_{11} \geq \frac{1}{\gamma_{11}}$. Lemma 2.1 shows that $\gamma_{11} = \frac{|G_{p'}|}{|G|_{p'}|G|_p} = \frac{|G_{p'}|}{|G|}$. Thus $c_{11} \geq \frac{|G|}{|G_{p'}|}$.

Suppose that $c_{11} = \frac{|G|}{|G_{p'}|}$. Since $c_{11}\gamma_{11} = 1$, $l(B_0) = 1$ by Corollary 3.2. Hence G is p-nilpotent, by ([15], Chap. V, Theorem 8.3). Since the converse is obvious, we are done. □

4. Some evidence for Conjecture 2.3

In this section we show some evidence for the conjecture.

Remark 4.1. Conjecture 2.3 has an affirmative answer if $l(B) = 1$. In this case the Cartan matrix of B is $C_B = p^d$ where d is the defect of B, since $\det C_B$ is the product of elementary divisors. Thus $\gamma_{\beta\beta} \cdot p^d = 1$ for $\mathrm{IBr}_p(B) = \{\beta\}$, and Conjecture 2.3 holds.

Proposition 4.2. *Let B be a p-block with a cyclic defect group. Then Conjecture 2.3 holds true.*

Proof. By ([4], Chap. VII, Lemma 10.11) we immediately get

$$l(B) = \min_{z \in \mathbb{Z}^{l(B)}} zp^d C^{-1} z^t \leq z_i p^d C^{-1} z_i^t = p^d \gamma_{ii},$$

where C is the Cartan matrix of B, d is the defect of B and $z_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the i-th position 1 and 0 elsewhere. $\square$

Note that the proof of Proposition 4.2 also shows that Murai's conjecture has an affirmative answer if the Sylow p-subgroup is cyclic.

Proposition 4.3. *If B is a 2-block of G having a dihedral, a semidihedral or a generalized quaternion group as defect group D, then Conjecture 2.3 holds true.*

Proof. Note that B is a block of tame representation type, and the Cartan matrices of such blocks are known by the classification of Erdmann [2]. In particular $l(B) \leq 3$. According to Remark 4.1 we may assume that $l(B) \geq 2$. Then the occurring matrices are listed in [8]. If $l(B) = 2$, then B has a Cartan matrix C of the form

$$\begin{pmatrix} 4k & 2k \\ 2k & k+r \end{pmatrix}$$

with natural numbers k and r, where $\{k, r\} = \{1, \frac{|D|}{4}\}$ or $\{k, r\} = \{2, \frac{|D|}{4}\}$. Note that $|D| \geq 8$, since a block with Klein four defect group cannot have two simple modules. We have $\det C = 4kr$. One easily computes $\gamma_{11} = \frac{k+r}{4kr}, \gamma_{22} = \frac{1}{r}$. Then in the first case, we have

$$\gamma_{11}|D| = k + r = 1 + \frac{|D|}{4} \geq 3 \text{ and } \gamma_{22}|D| = \frac{1}{r}|D| \geq 4,$$

and in the second case we have

$$\gamma_{11}|D| = \frac{k+r}{2} = 1 + \frac{|D|}{8} \geq 2 \text{ and } \gamma_{22}|D| = \frac{1}{r}|D| \geq 4.$$

Hence we are done for Cartan matrices of blocks B with $l(B) = 2$.

One of the Cartan matrices for $l(B) = 3$ is

$$C = \begin{pmatrix} 4k & 2k & 2k \\ 2k & k+a & k \\ 2k & k & k+a \end{pmatrix},$$

where $k = \frac{|D|}{4}$ and $a \in \{1, 2\}$. Then $\det C = a^2|D|$ and $\gamma_{11}|D| = \frac{2k+a}{a}$, $\gamma_{22}|D| = \gamma_{33}|D| = \frac{4k}{a}$. If $a = 1$ then

$$\gamma_{11}|D| = 2k + 1 \geq 5 \text{ and } \gamma_{22}|D| = \gamma_{33}|D| = 4k = |D| \geq 8,$$

and if $a = 2$ then

$$\gamma_{11}|D| = k + 1 \geq 3 \text{ and } \gamma_{22}|D| = \gamma_{33}|D| = 2k = \frac{|D|}{2} \geq 4.$$

The remaining cases listed in [8] can be handled in the same way. □

Remark 4.4. We do not intend to prove Conjecture 2.3 or Murai's conjecture for p-blocks of p-solvable groups, since both of their proofs seem more difficult than that of the famous $k(GV)$-problem (which consists of the work of a series of authors, and was verified affirmatively, but needed a period of more than forty years (see [5])).

5. Murai's conjecture for S_n and A_n

In this section, we prove that Murai's conjecture holds true for symmetric and alternating groups. We start with a result of Babai, Pálfy and Saxl on the proportion of p-regular elements in the alternating group A_n.

Theorem 5.1. *Let p be a prime number, $n \geq 3$ an integer and $w = \lfloor n/p \rfloor$. Then the proportion of p-regular elements in the alternating group A_n is given by the following formulas:*

(*a*) *if $p = 2$:*

$$2\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{2p}\right)\cdots\left(1 - \frac{1}{wp}\right);$$

(*b*) *if $p > 2$ and $n \equiv 0$ or $1 \pmod p$:*

$$\begin{aligned} &\left(1 - \tfrac{1}{p}\right)\left(1 - \tfrac{1}{2p}\right)\cdots\left(1 - \tfrac{1}{wp}\right) \\ + \quad &\tfrac{(-1)^w}{wp}\left(1 + \tfrac{1}{p}\right)\left(1 + \tfrac{1}{2p}\right)\cdots\left(1 + \tfrac{1}{(w-1)p}\right); \end{aligned}$$

(c) *if $p > 2$ and $n \not\equiv 0$ or $1 \pmod p$:*

$$\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{2p}\right)\cdots\left(1 - \frac{1}{wp}\right).$$

Proof. This is ([1], Theorem 2.1). □

For integers $s, t \geq 1$ let $k(s, t)$ be the number of s-tuples $(\lambda_1, \ldots, \lambda_s)$ of partitions λ_i such that $\sum_{i=1}^{s} |\lambda_i| = t$. In particular, $k(1, t)$ is the number of partitions of t.

Lemma 5.2 (Olsson). *Let $s, t \geq 1$. Then $k(s,t) < (s+1)^t$. If moreover $s \geq 2$, then $k(s,t) \leq s^t$ unless $s = 2$ and $t \leq 6$.*

Proof. This is ([12], Lemma 5.1). □

Lemma 5.3. *Let p be an odd prime number and $w' \geq 2$. Then $p^{w'-1} \geq 6w'$ unless*
(i) $p = 5, 7, 11$ and $w' = 2$, or
(ii) $p = 3$ and $w' = 2, 3$.

Proof. Suppose that $p \geq 13$. For $w' = 2$, it is clear that $p^{w'-1} \geq 13 > 12 = 6w'$. By induction on w', we have

$$p^{w'-1} = p \cdot p^{w'-2} \geq p \cdot 6(w'-1) > 6w',$$

and so the lemma holds for $p \geq 13$ and $w' \geq 2$. For either $p = 5, 7, 11$ and $w' \geq 3$, or $p = 3$ and $w' \geq 4$, the lemma similarly holds by induction on w', which finishes the proof. □

Proposition 5.4. *Let G be the symmetric group S_n or the alternating group A_n. Then G satisfies Murai's conjecture for any prime p.*

Proof. Denote by B_0 the principal p-block of G. Write $n = wp + r$ with $0 \leq r < p$. We may assume that $n \geq 5$, since for $n \leq 4$ the assertion is well known to be true, and can be verified easily.

We first let $G = S_n$. In this case, by ([18], Proposition 11.14) we have $\ell(B_0) = k(p-1, w)$. Note that the proportion of p-regular elements in G has been obtained by Erdős and Turán ([3], Lemma I) as

$$\frac{|G_{p'}|}{|G|} = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{2p}\right)\cdots\left(1 - \frac{1}{wp}\right).$$

Furthermore we have

$$|G|_p = p^{\lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots} \geq p^w \cdot p^{\lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \cdots}.$$

Hence, for $p > 3$ or $p = 3$ and $w > 6$, we have

$$\begin{array}{rcll} m_p(G) = \frac{|G_{p'}|}{|G|} \cdot |G|_p & \geq & \left(1 - \frac{1}{p}\right)\cdots\left(1 - \frac{1}{wp}\right) \cdot p^w & \\ & = & (p-1)\left(p - \frac{1}{2}\right)\cdots\left(p - \frac{1}{w}\right) & \\ & \geq & (p-1)^w & \\ & \geq & k(p-1, w) & \text{(by Lemma 5.2)} \\ & = & l(B_0). & \end{array}$$

Similarly, for $p = 2$ and $w \geq 4$, we have

$$\begin{array}{rcll} m_2(G) & \geq & \left[\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\cdots\left(1 - \frac{1}{2w}\right) \cdot 2^w\right] \cdot 2^{\lfloor \frac{n}{2^2} \rfloor + \lfloor \frac{n}{2^3} \rfloor} & (*) \\ & \geq & \left(\frac{3}{2}\right)^{w-1} \cdot 2^{\frac{w}{2}+1} & \\ & > & 2^w \geq k(1, w) = l(B_0). & \end{array}$$

The small cases where either $p = 2$ and $w \le 3$ or $p = 3$ and $w \le 6$ can be checked directly with MOC [13] and the formula $(*)$

$$m_p(S_n) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{2p}\right)\cdots\left(1 - \frac{1}{wp}\right)|S_n|_p.$$

Doing this, note that $m_p(S_n)$ and $l(B_0)$ do not depend on the rest $r = n - pw$. Consequently we only have to check the cases
(1) $p = 2$, $w \le 3$ and $n = 6$ and
(2) $p = 3$, $w \le 6$ and $n = 6, 9, 12, 15$ and 18.
In the case (1) we have $l(B_0) = 3 < m_2(S_6) = 5$.
In the cases (2) we obtain for
$n = 6$: $l(B_0) = 5 = m_3(S_6)$,
$n = 9$: $l(B_0) = 10 < 40 = m_3(S_9)$,
$n = 12$: $l(B_0) = 20 < 110 = m_3(S_{12})$,
$n = 15$: $l(B_0) = 36 < 308 = m_3(S_{15})$,
$n = 18$: $l(B_0) = 65 < 2618 = m_3(S_{18})$.

We now let $G = A_n$. It is well known that any p-block of S_n is parameterized by its p-core (i.e., the p-core of a partition of n corresponding to an irreducible character of the block) and its weight (see [18]). We write $\widetilde{B}_0$ for the principal p-block of S_n and $\mu(\widetilde{B}_0)$ for the p-core of $\widetilde{B}_0$.

Suppose that $p = 2$. By Theorem 5.1, we get $m_2(A_n) = m_2(S_n)$. If we assume that $n \ge 16$, then by adding the factor $\lfloor \frac{n}{2^4} \rfloor$ in the above formula $(*)$ we get

$$m_2(A_n) \ge 2^{w+1} \ge 2(k(1,w)) \ge l(B_0),$$

since by [18, Proposition 12.9], we have

$$l(B_0) = \begin{cases} k(1,w) & \text{if } w \text{ is odd} \\ k(1,w) + k(1,w') & \text{if } w = 2w'. \end{cases}$$

It remains to check seperately the cases $n = 6, 8, 10, 12$ and 14. Note that $l(B_0(A_{2m})) = l(B_0(A_{2m+1}))$ and $m_2(A_{2m}) = m_2(A_{2m+1})$. Here we get for
$n = 6:$ $l(B_0) = 3$ and $m_2(A_6) = 5$,
$n = 8:$ $l(B_0) = 7$ and $m_2(A_8) = 35$,
$n = 10:$ $l(B_0) = 7$ and $m_2(A_{10}) = 63$,
$n = 12:$ $l(B_0) = 14$ and $m_2(A_{12}) = 231$,
$n = 14:$ $l(B_0) = 15$ and $m_2(A_{14}) = 429$.

Finally, we suppose that p is odd. If $\mu(\widetilde{B}_0)$ is not self-conjugate, then $n \not\equiv 0$ or $1 \pmod p$ and $l(B_0) = l(\widetilde{B}_0) = k(p-1, w)$ by ([18], Proposition 12.8 (i)). Furthermore, the proportion of p-regular elements is the same as for the corresponding symmetric group, by Theorem 5.1 (c). Thus the result follows as for S_n.

So we may assume that $\mu(\widetilde{B}_0)$ is self-conjugate and the weight w of $\widetilde{B}_0$ is positive. In particular, $n \equiv 0$ or $1 \pmod{p}$. By ([17], Proposition 2.13) or ([18], Proposition 12.8 (ii)), we get

$$l(B_0) = \begin{cases} \frac{1}{2}k(p-1,w) & \text{if } w \text{ is odd,} \\ \frac{1}{2}\left(k(p-1,w) + 3k\left(\frac{1}{2}(p-1), w'\right)\right) & \text{if } w = 2w'. \end{cases}$$

We first suppose that w is odd. Since $(p-1)(2p-1)\cdots(wp-1) \geq 2(p+1)(2p+1)\cdots((w-1)p+1)$, we have

$$\frac{1}{2}\left(1-\frac{1}{p}\right)\cdots\left(1-\frac{1}{wp}\right) \geq \frac{1}{wp}\left(1+\frac{1}{p}\right)\cdots\left(1+\frac{1}{(w-1)p}\right).$$

Hence, by Theorem 5.1 (b), we get similar as for S_n

$$\begin{aligned} m_p(G) &\geq \frac{1}{2}\left(1-\frac{1}{p}\right)\cdots\left(1-\frac{1}{wp}\right)\cdot p^w \\ &\geq \frac{1}{2}(p-1)^w \\ &\geq \frac{1}{2}k(p-1,w) = l(B_0) \qquad \text{(by the latter part of Lemma 5.2)} \end{aligned}$$

except possibly $p = 3$ and $w = 3, 5$. For these cases we have
$n = 9, 10$: $l(B_0) = 5$ and $m_3(A_9) = m_3(A_{10}) = 26$,
$n = 15, 16$: $l(B_0) = 18$ and $m_3(A_{15}) = m_3(A_{16}) = 217$.

Thus we are left with the case that w is even. If $w = 2$, then by Theorem 5.1 (b),

$$\begin{aligned} m_p(G) &\geq \left[\left(1-\frac{1}{p}\right)\left(1-\frac{1}{2p}\right) + \frac{1}{2p}\left(1+\frac{1}{p}\right)\right]\cdot p^2 \\ &= (p-1)\left(p-\frac{1}{2}\right) + \frac{1}{2}(p+1) = p^2 - p + 1 \end{aligned}$$

and

$$\begin{aligned} l(B_0) &= \frac{1}{2}\left[k(p-1,2) + 3k\left(\frac{p-1}{2}, 1\right)\right] \\ &= \frac{1}{2}\left[2(p-1) + \frac{(p-1)(p-2)}{2}\right] + \frac{3}{4}(p-1) \qquad \text{(by [18, (3.11)])} \\ &= \frac{p^2}{4} + p - \frac{5}{4}. \end{aligned}$$

Hence we obtain $m_p(G) \geq l(B_0)$.

So we may finally assume that $w = 2w' \geq 4$. By Theorem 5.1 (b), the proportion $\frac{|(A_n)_{p'}|}{|A_n|}$ of p-regular elements in the alternating group A_n is

$$\left(1-\frac{1}{p}\right)\left(1-\frac{1}{2p}\right)\cdots\left(1-\frac{1}{wp}\right)+\frac{1}{wp}\left(1+\frac{1}{p}\right)\left(1+\frac{1}{2p}\right)\cdots\left(1+\frac{1}{(w-1)p}\right).$$

Since $|A_n|_p = |S_n|_p \geq p^w$, we get as for S_n

$$m_p(A_n) \geq k(p-1,w) + \frac{1}{w}(p+1)(p+\frac{1}{2})\cdots(p+\frac{1}{w-1}).$$

So we are done if

$$\frac{1}{w}(p+1)(p+\frac{1}{2})\cdots(p+\frac{1}{w-1}) \geq 3k(\frac{p-1}{2},w').$$

According to Lemma 5.2 we have $\left(\frac{p+1}{2}\right)^{w'} \geq k(\frac{p-1}{2},w')$. Also, by Lemma 5.3, we have $p^{w'-1} \geq 6w'$ and so

$$\frac{1}{w}(p+1)\cdots(p+\frac{1}{w-1}) \geq \frac{1}{2w'}p^{2w'-1} \geq 3\left(\frac{p+1}{2}\right)^{w'} \geq 3k(\frac{p-1}{2},w')$$

unless (i) $p = 5, 7, 11$ and $w' = 2$; or (ii) $p = 3$ and $w' = 2, 3$.

For the possible exceptions $p = 5, 7, 11$ and $w' = 2$, we also have

$$\begin{array}{rcl} \frac{1}{w}(p+1)\cdots(p+\frac{1}{w-1}) & = & \frac{1}{4}(p+1)\cdot(p+\frac{1}{2})\cdot(p+\frac{1}{3}) \\ & > & 3(p-1)+\frac{3(p-1)(p-3)}{8} \\ & = & 3k(\frac{p-1}{2},2) \\ & = & 3k(\frac{p-1}{2},w'), \end{array}$$

and so we are done in this case. For $p = 3$ and $w' = 2, 3$, i.e., $n = 12, 13, 18$ and 19, we get
$n = 12, 13:$ $l(B_0) = 13$ and $m_3(A_{12}) = m_3(A_{13}) = 145$
$n = 18, 19:$ $l(B_0) = 37$ and $m_3(A_{18}) = m_3(A_{19}) = 3346$,
which completes the proof. □

Remark 5.5. In ([12], Proposition 5.2) Malle and Robinson proved $l(B) \leq p^w$ in the case that B is a p-block of a symmetric group, an alternating group or their covering groups and w is the weight of B. If B_0 is the principal 2-block of S_n ($n \leq 7$), then $m_2(S_n) \leq p^w$. But $m_2(S_8) = 35 > p^w = 2^4 = 16$. If B_0 is the principal 3-block of S_n ($n \leq 8$), then $m_3(S_n) \leq p^w$, but $m_3(S_9) = 40 > p^w = 3^3 = 27$.

Acknowledgements. We are deeply grateful to the referee for his/her invaluable comments.

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