The conjecture of Wei and Yang on the weight hierarchy of product codes

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Abstract-**Let V and W be codes and let $C = V \otimes W$ be the product code of V and W. In [3], Wei and Yang conjectured a formula for the generalized Hamming weights of $V \otimes W$ in terms of those of V and W provided that both V and W satisfy the chain condition. The main aim of this paper is to prove the conjecture. Furthermore we generalize the formula to a product code with more than two components.**

Index Terms-**Product code, weight hierarchy, chain condition.**

I. INTRODUCTION

Throughout the paper all codes are linear and defined over the finite field K.
For $v = (x_1, ..., x_n) \in K^n$ we define the support of v by

$$\text{supp } v = \{i \mid x_i \neq 0\}$$

and for $X \subseteq K^n$ we put

$$\text{supp } X = \bigcup_{v \in X} \text{supp } v.$$

The weight hierarchy of an $[n, k, d]$ code C is the sequence $(d_1, ..., d_k)$ where

$$d_r = \min\{|\operatorname{supp} D \mid \mid D \leq C, \dim D = r\}.$$

In particular $d_1 = d$. For convenience we also set $d_0 = 0$. One says that C satisfies the chain condition if there exists a sequence

$$C_1 < C_2 < \cdots < C_k = C$$

*This work was done while the author was visiting the University of Magdeburg. She wishes to thank the Department of Mathematics for his kind hospitality and the Caja de Ahorros de la Inmaculada for financial support.

of subspaces C_i of C with $\mid \mathrm{supp}\, C_r \mid = d_r$ for $r = 1, ..., k$. Note that many of the well-known codes satisfy this condition, for instance Hamming codes, Simplex codes, Reed-Muller codes, MDS codes, extended Golay codes (see [3]).

Now let V be an $[n, k]$ and let W be an $[m, l]$ code. The product code C of V and W is defined as the tensor product $V \otimes W$. If

$$(y_1, ..., y_m)^t (x_1, ..., x_n) = \begin{pmatrix} y_1 \\ . \\ . \\ . \\ y_m \end{pmatrix} \begin{pmatrix} x_1 & . & . & . & x_n \end{pmatrix} = (y_i x_j)_{\substack{\mathrm{i=1,..,m} \\ \mathrm{j=1,..,n}}}$$

denotes the dyadic product of $w = (y_1, ..., y_m)$ and $v = (x_1, ..., x_n)$, then C can be represented by

$$C = < w^t v \mid v \in V, w \in W >,$$

i.e. the elements of C are matrices of type (m, n) where the rows are elements of V and the columns are elements of W.

The problem which we are interested in is the following: *Can we compute the weight hierarchy of C from the knowledge of the weight hierarchies of V and W?*

In order to attack this problem Wei and Yang defined in [3]

$$d_r^*(V \otimes W) = \min \Sigma_{i=1}^k (d_i(V) - d_{i-1}(V)) d_{t_i}(W)$$

where the minimum is taken over all sequences (t_i) with $0 \leq t_k \leq ... \leq t_2 \leq t_1 \leq l$ and $\Sigma_{i=1}^k t_i = r \leq kl$.

In the same paper the authors proved that

$$d_r(V \otimes W) \leq d_r^*(V \otimes W) \tag{1}$$

for all $r = 1, ..., kl$ and they conjectured that even equality holds true if both V and W satisfy the chain condition.

For small r's, i.e. $r \leq 4$, the conjecture has been verified by Barbero and Tena in [1].

The main aim of this paper is to prove that

$$d_r^*(V \otimes W) \leq d_r(V \otimes W) \tag{2}$$

for all $r = 1, ..., kl$ provided V satisfies the chain condition. As a consequence the conjecture of Wei and Yang has a positive answer, i.e. $d_r(V \otimes W) = d_r^*(V \otimes W)$ if both V and W satisfy the chain condition. Furthermore, in [2], Helleseth and Kløve computed d_r^* for the tensor product of some codes which satisfy the chain condition. Thus their formulas for d_r^* describe exactly the weight distribution.

Easy examples show that in general $d_r^*(V \otimes W) \neq d_r(V \otimes W)$ if one of the codes fails to satisfy the chain condition.

In the last section we generalize the Wei-Yang formula to a product code with more than two components.

II. PROOF OF THE WEI-YANG CONJECTURE

Let $C = V \otimes W =< w^t v \mid v \in V, w \in W >$ where V is an $[n, k]$ code and W is an $[m, l]$ code.

For $x \in C$ we denote by $\pi_j^C(x)$ $(j = 1, ..., n)$ the j-th column and by $\pi_i^R(x)$ $(i = 1, ..., m)$ the i-th row of x. If D is a subset of C we put

$$\pi_j^C(D) = \{\pi_j^C(x) \mid x \in D\} \subseteq W \text{ and}$$
$$\pi_i^R(D) = \{\pi_i^R(x) \mid x \in D\} \subseteq V.$$

Furthermore, let

$$\rho_j^C(D) = \{x \in D \mid \pi_j^C(x) = 0\} \text{ and}$$
$$\rho_i^R(D) = \{x \in D \mid \pi_i^R(x) = 0\}.$$

For any vector v and $1 \leq s \leq \text{length}(v)$ we denote by $v_{(s)}$ the s-th coordinate of v.

Lemma 1. Let $C = V \otimes W$. If

$$W_{(s)} = \{w = (y_1, ..., y_m) \in W \mid y_s = 0\},$$

then

$$\rho_s^R(C) = V \otimes W_{(s)}.$$

Proof. If $W_{(s)} = W$, the assertion is obvious. So we may assume that $W =< w > \oplus W_{(s)}$ for some $w \in W \setminus W_{(s)}$. Let $x \in \rho_s^R(C)$. Since $V \otimes W = V \otimes < w > \oplus V \otimes W_{(s)}$, we may write $x = y + z$ with $y = v \otimes w$ $(v \in V)$ and $z \in V \otimes W_{(s)}$. Obviously $\pi_s^R(V \otimes W_{(s)}) = 0$, hence

$$0 = \pi_s^R(x) = \pi_s^R(y) \oplus \pi_s^R(z) = \pi_s^R(y) = \pi_s^R(v \otimes w).$$

Now, as $w \notin W_{(s)}$ we obtain $v = 0$, therefore $x = z \in V \otimes W_{(s)}$ and the assertion of the Lemma follows since $V \otimes W_{(s)} \leq \rho_s^R(C)$. $\square$

From now on we suppose that V satisfies the chain condition, i.e. there exists a chain $V_1 < V_2 < ... < V_k = V$ such that $d_r(V) = \mid \operatorname{supp} V_r \mid$ for $r = 1, .., k$. We put $V_0 = 0$.

Proposition 2. Let $D \leq V \otimes W$. Then there exists a basis of W, say $\mathbf{W} = \{w_1, ..., w_l\}$ and integers $0 \leq \alpha_i \leq k$ $(i = 1, .., l)$ such that for $\bar{D} = V_{\alpha_1} \otimes w_1 \oplus ... \oplus V_{\alpha_l} \otimes w_l$ the following conditions hold.

$$(i) \quad \dim D = \dim \bar{D} \text{ and}$$

$$(ii) \quad \mid \operatorname{supp} \pi_i^R(\bar{D}) \mid \leq \mid \operatorname{supp} \pi_i^R(D) \mid \text{ for } 1 \leq i \leq m.$$

In particular, we have $\mid \operatorname{supp} \bar{D} \mid \leq \mid \operatorname{supp} D \mid$.

Proof. We argue by induction on l. For $l = 1$ the assertion is obvious. Now we put $r_i = \dim \pi_i^R(D)$ and determine s and r_{i_j} for $1 \leq j \leq s \leq n$ such that

$$r_{i_1} = r_{i_2} = ... = r_{i_{s-1}} = 0,$$
$$r_{i_s} \neq 0 \text{ and}$$
$$r_{i_s} \leq r_t \text{ for } t \in \{1, ..., n\} \setminus \{i_1, i_2, ..., i_{s-1}\}.$$

Without lost of generality we may assume that $i_j = j$.
Since $r_s \neq 0$ and $\dim D = \dim \pi_s^R(D) + \dim \rho_s^R(D) = r_s + \dim \rho_s^R(D)$, we have $\rho_s^R(D) \subset D$. Thus there exists $x \in D$ such that $\pi_s^R(x) \neq 0$. Hence there is a $j \in \{1, ..., n\}$ such that $0 \neq (\pi_j^C(x))_{(s)}$ which is the s-th coordinate of the column vector $\pi_j^C(x)$.
Let $w = \pi_j^C(x)$. Clearly, $w \in W \setminus W_{(s)}$ and therefore $W_{(s)} < W$. By Lemma 1, we have $\rho_s^R(C) = V \otimes W_{(s)}$ and we may apply the inductive hypothesis to

$$\rho_s^R(D) \leq \rho_s^R(C) = V \otimes W_{(s)}.$$

We deduce that there exists a basis $\mathbf{W}_{(s)} = \{w_2, ..., w_l\}$ of $W_{(s)}$ and integers $0 \leq \alpha_i \leq k$ $(i = 2, ..., l)$ such that for

$$\bar{D}_1 = V_{\alpha_2} \otimes w_2 \oplus ... \oplus V_{\alpha_l} \otimes w_l$$

we have $\dim \bar{D}_1 = \dim \rho_s^R(D)$ and

$$\mid \operatorname{supp} \pi_i^R(\bar{D}_1) \mid \leq \mid \operatorname{supp} \pi_i^R(\rho_s^R(D)) \mid$$

for all $i = 1, ..., n$.

If $w_1 = w$ which we have fixed above ($w_1 \in W \setminus W_{(s)}$), then $\{w_1, w_2, ..., w_l\}$ is a basis of W. We put $\alpha_1 = r_s = \dim \pi_s^R(D)$ and

$$\bar{D} = V_{r_s} \otimes w \oplus \bar{D}_1 = V_{\alpha_1} \otimes w_1 \oplus \bar{D}_1.$$

Thus $\bar{D}$ has the shape as desired and

$$\dim \bar{D} = \dim V_{r_s} + \dim \bar{D}_1 = r_s + \dim \rho_s^R(D) =$$
$$= \dim \pi_s^R(D) + \dim \rho_s^R(D) = \dim D.$$

It remains to show that

$$| \operatorname{supp} \pi_i^R(\bar{D}) | \leq | \operatorname{supp} \pi_i^R(D) | \quad \text{for } 1 \leq i \leq m.$$

To prove this we use

$$\pi_i^R(\bar{D}) = \pi_i^R(V_{r_s} \otimes w) + \pi_i^R(\bar{D}_1)$$

and distinguish three cases.

Case $1 \leq i < s$: Since $w = \pi_j^C(x)$ and $\dim \pi_i^R(D) = 0$, we have

$$w_{(i)} = (\pi_j^C(x))_{(i)} = (\pi_i^R(x))_{(j)} = 0.$$

Thus $\pi_i^R(V_{r_s} \otimes w) = 0$. As

$$0 \leq | \operatorname{supp} \pi_i^R(\bar{D}_1) | \leq | \operatorname{supp} \pi_i^R(\rho_s^R(D)) | \leq | \operatorname{supp} \pi_i^R(D) | = 0,$$

we are done.

Case $i = s$: Since $\bar{D}_1 \leq \rho_s^R(C)$, we have $\pi_s^R(\bar{D}_1) = 0$. Furthermore, since $w_{(s)} \neq 0$, we get $\pi_s^R(\bar{D}) = \pi_s^R(V_{r_s} \otimes w) = V_{r_s}$. Note that $\dim \pi_s^R(D) = r_s$, hence

$$| \operatorname{supp} \pi_s^R(D) | \geq d_{r_s}(V) = | \operatorname{supp} V_{r_s} | = | \operatorname{supp} \pi_s^R(\bar{D}) | .$$

Case $s < i$: Observe that

$$\pi_i^R(\bar{D}) = \pi_i^R(V_{r_s} \otimes w) + \pi_i^R(\bar{D}_1)$$
$$= \pi_i^R(V_{\alpha_1} \otimes w_1) + \pi_i^R(V_{\alpha_2} \otimes w_2) + ... + \pi_i^R(V_{\alpha_l} \otimes w_l).$$

Since the spaces V_{α_i} are members of the chain, which defines the chain condition for V, one easily sees that

$$| \operatorname{supp} \pi_i^R(\bar{D}) | = \max \{ | \operatorname{supp} \pi_i^R(V_{r_s} \otimes w) |, | \operatorname{supp} \pi_i^R(\bar{D}_1) | \}.$$

By the inductive hypothesis we have

$$| \operatorname{supp} \pi_i^R(D) | \geq | \operatorname{supp} \pi_i^R(\rho_s^R(D)) | \geq | \operatorname{supp} \pi_i^R(\bar{D}_1) |,$$

and since $\dim \pi_i^R(D) = r_i \geq r_s$, we also have

$$| \operatorname{supp} \pi_i^R(D) | \geq d_{r_s}(V) \geq | \operatorname{supp} \pi_i^R(V_{r_s} \otimes w) | \, .$$

From this we deduce

$$| \operatorname{supp} \pi_i^R(D) | \geq | \operatorname{supp} \pi_i^R(\bar{D}) | \, .$$

So far we have proved (i) and (ii). Finally note that for any $D \leq C = V \otimes W$ the support of D can be computed by looking at the rows or the columns of the matrices in D; more precisely, we have

$$| \operatorname{supp} D | = \Sigma_{i=1}^m \, | \operatorname{supp} \pi_i^R(D) | = \Sigma_{j=1}^n \, | \operatorname{supp} \pi_j^C(D) | \, .$$

Thus we also get

$$| \operatorname{supp} \bar{D} | \leq | \operatorname{supp} D | \, .$$

$\square$

Lemma 3. Let $D \leq C = V \otimes W$ with $\dim D = r$. Then there exists a sequence $l \geq i(1) \geq i(2) \geq ... \geq i(k) \geq 0$ of integers $i(s)$ with $\Sigma_{s=1}^k i(s) = r$ such that

$$| \operatorname{supp} D | \geq \Sigma_{s=1}^k (d_s(V) - d_{s-1}(V)) d_{i(s)}(W).$$

Proof. By Proposition 2, we may assume that for a suitable basis $\mathbf{W} = \{w_1, ..., w_l\}$ of W and integers $0 \leq \alpha_i \leq k$ $(i = 1, ..., l)$ the space D is of the form

$$D = V_{\alpha_1} \otimes w_1 \oplus ... \oplus V_{\alpha_l} \otimes w_l.$$

Let $\mathbf{V} = \{v_1, ..., v_k\}$ be a basis of V such that $V_i = < v_1, ..., v_i >$ and $d_i(V_i) = | \operatorname{supp} V_i |$.
For $1 \leq s \leq k$ we put

$$D_s^W = < w_i \in \mathbf{W} \mid v_s \otimes w_i \in D >$$

and $i(s) = \dim D_s^W$.
Clearly, $\dim D = \Sigma_{s=1}^k i(s)$.
We claim that $i(s) \leq i(s-1)$ for $2 \leq s \leq k$. In order to see that let $\beta(s) = \{i \mid w_i \in D_s^W\}$.
Note first that $| \beta(s) | = \dim D_s^W = i(s)$. For $i \in \beta(s)$ we have

$$v_s \otimes w_i \in D = V_{\alpha_1} \otimes w_1 \oplus ... \oplus V_{\alpha_l} \otimes w_l,$$

hence $s \le \alpha_i$. But then $v_{s-1} \otimes w_i \in V_{\alpha_i} \otimes w_i \le D$ which shows that $i \in \beta(s-1)$ and the claim follows.

Next observe that $d_s(V) - d_{s-1}(V)$ is the cardinality of the set

$$\alpha(s) = \{1 \le j \le n \mid (v_s)_{(j)} \ne 0, (v_t)_{(j)} = 0 \text{ for } 1 \le t < s\}.$$

For $j \in \alpha(s)$ we have

$$D_s^W = \pi_j^C(v_s \otimes D_s^W) \le \pi_j^C(D).$$

By computing the support of D via the columns we finally obtain

$$\begin{aligned}
\mid \operatorname{supp} D \mid = \Sigma_{j=1}^n \mid \operatorname{supp} \pi_j^C(D) \mid &= \Sigma_{s=1}^k \Sigma_{j\in\alpha(s)} \mid \operatorname{supp} \pi_j^C(D) \mid \ge \\
\ge \Sigma_{s=1}^k \Sigma_{j\in\alpha(s)} \mid \operatorname{supp} D_s^W \mid &\ge \Sigma_{s=1}^k \Sigma_{j\in\alpha(s)} d_{i(s)}(W) = \\
&= \Sigma_{s=1}^k (d_s(V) - d_{s-1}(V)) d_{i(s)}(W).
\end{aligned}$$

□

If we put

$$d_r^*(V \otimes W) = \min\ \Sigma_{i=1}^k (d_i(V) - d_{i-1}(V)) d_{t_i}(W)$$

where the minimum is taken over all sequences (t_i) with $0 \le t_k \le ... \le t_2 \le t_1 \le l$ and $\Sigma_{i=1}^k t_i = r \le (\dim V)(\dim W)$, we obtain from Lemma 3

Theorem 4. Suppose that V satisfies the chain condition. Then we have

$$d_r(V \otimes W) \ge d_r^*(V \otimes W)$$

for all $r = 1, ..., (\dim V)(\dim W)$.

In [3], Wei and Yang proved that $d_r(V \otimes W) \le d_r^*(V \otimes W)$ provided both V and W satisfy the chain condition. Thus as a consequence of the Theorem we get

Corollary 5. If both V and W satisfy the chain condition, then

$$d_r(V \otimes W) = d_r^*(V \otimes W)$$

for all $r = 1, ..., (\dim V)(\dim W)$.

If only one of the codes satisfy the chain condition, we do not have equality in general as the next example shows.

Example 6. Let $F = \{0, 1, \psi, \psi^2\} = \mathbb{F}_4$ be a field with four elements.

Let V be the $[5,2,3]_4$ code defined by the generator matrix

$$G_V = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and let W be the $[6,3,2]_4$ code defined by

$$G_W = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \psi & \psi^2 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Clearly, V satisfies the chain condition and has hierarchy $d_1(V) = 3$, $d_2(V) = 5$. The code W has hierarchy $d_1(W) = 2$, $d_2(W) = 4$, $d_3(W) = 6$ and does not satisfy the chain condition.

Now, $d_3^*(V \otimes W) = \min\{3d_{i(1)}(W) + 2d_{i(2)}(W) \mid 0 \leq i(2) \leq i(1) \leq 3, i(1) + i(2) = 3\} = \min\{18, 16\} = 16$.

We claim that $d_3(V \otimes W) = 18$. Let D be a subspace of $V \otimes W$ of dimension 3 with $|\operatorname{supp} D| = d_3(V \otimes W)$. By Proposition 2, we may suppose that either

(i) $$D = V_1 \otimes x_1 \oplus V_1 \otimes x_2 \oplus V_1 \otimes x_3 = V_1 \otimes W$$

with $< x_1, x_2, x_3 >= W$ or

(ii) $$D = V_1 \otimes x_1 \oplus V_2 \otimes x_2$$

where $< x_1, x_2 >$ is a 2-dimensional subspace of W.

In the first case one easily sees that $|\operatorname{supp} D| = 18$. To analyse the second case, let w_1, w_2, w_3 be the three rows of G_W.

If $< x_2 >=< w_1 >$, then $|\operatorname{supp}(V_2 \otimes x_2)| = 10$. Since $x_1 \notin < w_1 >$ and $d(< w_2, w_3 >) = 3$, we get $|\operatorname{supp} D| \geq 10 + 9 = 19$.

Thus we may suppose that $< x_2 > \neq < w_1 >$. Clearly, $wt(x_2) \geq 3$. If $wt(x_2) \geq 4$, then $|\operatorname{supp} D| \geq |\operatorname{supp}(V_2 \otimes x_2)| \geq 20$. Hence we may suppose that $wt(x_2) = 3$. This forces $x_2 \in < w_2, w_3 >$ and $|\operatorname{supp}(V_2 \otimes x_2)| = 15$. If $x_1 \notin < w_2, w_3 >$, then $|\operatorname{supp} D| \geq 15+6 = 21$. It remains $x_1 \in < w_2, w_3 >$ to consider. Now $< x_1, x_2 >=< w_2, w_3 >$ and $|\operatorname{supp} D| \geq 18$.

The next example shows that in general $V \otimes W$ does not necessarily satify the chain condition, even if both V and W do.

Example 7. Let $F = \{0, 1, \psi, \psi^2\} = \mathbb{F}_4$.
Let V be the code generated by

$$G_V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and let W be generated by

$$G_W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \psi & \psi^2 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Obviously, both V and W satisfy the chain condition. We claim that $C = V \otimes W$ fails to satisfy the chain condition. Let v_1, v_2 be the rows of G_V and let w_1, w_2, w_3 be those of G_W.
Note that there is only one subspace $C_1 \leq C$ of dimension 1 with $\mid \operatorname{supp} C_1 \mid = 1$, namely $C_1 = < v_1 \otimes w_1 >$.
It is also easy to check that there is only one subspace $C_2 \leq C$ of dimension 2 such that $\mid \operatorname{supp} C_2 \mid \leq 3$, namely $C_2 = V \otimes w_1$, which actually has support length 3.
Now let $C_3 \leq C$ of dimension 3 and with $d_3(C) = \mid \operatorname{supp} C_3 \mid$. By the Theorem above, we know that $d_3(C) = 5$.
If $C_3 \geq C_2 = V \otimes w_1$, then $C_3 = V \otimes w_1 \oplus C'$ with $C' \leq V \otimes < w_2, w_3 >$. But $\mid \operatorname{supp} C_3 \mid \geq 3 + 3 = 6$, a contradiction. Thus C does not satify the chain condition. By the way, $v_1 \otimes W$ has support length 5.

III. GENERAL PRODUCT CODES

The formula given above to compute the generalized Hamming weights of a product of two codes can be generalized to a product code with more than two components. To do that, we need the following lemma.

Lemma 8. Assume that V satisfies the chain condition with $V_1 < ... < V_k = V$ and $V_0 = 0$. Let $D_s^V \leq V$ $(s \in \{1, ..., l\})$ and let

$$D = D_1^V \otimes w_1 \oplus D_2^V \otimes w_2 \oplus ... \oplus D_l^V \otimes w_l$$

with w_i $(i \in \{1, ..., l\})$ in the code W. If we put $i(s) = \dim D_s^V$ and

$$\bar{D} = V_{i(1)} \otimes w_1 \oplus V_{i(2)} \otimes w_2 \oplus ... \oplus V_{i(l)} \otimes w_l,$$

then

$$| \operatorname{supp} D | \geq | \operatorname{supp} \bar{D} | .$$

Proof. We compute $| \operatorname{supp} D |$ and $| \operatorname{supp} \bar{D} |$ via the rows. For $j \in \{1, ..., m\}$ fixed we obtain

$$\pi_j^R(D) = \Sigma_{s=1}^l (w_s)_{(j)} D_s^V \text{ and}$$
$$\pi_j^R(\bar{D}) = \Sigma_{s=1}^l (w_s)_{(j)} V_{i(s)}.$$

Now choose $1 \leq t \leq l$ such that

$$i(t) = \max \{i(s) \mid (w_s)_{(j)} \neq 0\}.$$

Then, since the spaces $V_{i(s)}$ are members of the chain, we get

$$| \operatorname{supp} \pi_j^R(\bar{D}) | = | \operatorname{supp} V_{i(t)} | = d_{i(t)}(V).$$

But we also have $D_t^V \leq \pi_j^R(D)$ and $\dim D_t^V = i(t)$. So we conclude that

$$| \operatorname{supp} \pi_j^R(D) | \geq | \operatorname{supp} D_t^V | \geq d_{i(t)}(V) = | \operatorname{supp} \pi_j^R(\bar{D}) |,$$

which yields the assertion of the Lemma. □

From now on we consider $[n_i, k_i]$ codes V^i $(i \in \{1, ..., t\})$ and assume that all of them satisfy the chain condition. Let $\mathbf{V}^i = \{v_1^i, ..., v_{k_i}^i\}$ be a basis of V^i such that

$$| \operatorname{supp} < v_1^i, ..., v_r^i > | = d_r(V^i) \quad (i \in \{1, ..., t\}, r \in \{1, ..., k_i\}).$$

Definition 9. Let $D \leq V^1 \otimes V^2 \otimes ... \otimes V^t$ be generated by elements of $\mathbf{V}^1 \otimes \mathbf{V}^2 \otimes ... \otimes \mathbf{V}^t$. We say that D is normal if it satisfies the following property.

Whenever $$v_{i_1}^1 \otimes v_{i_2}^2 \otimes ... \otimes v_{i_t}^t \in D$$

and $0 \leq j_l \leq i_l$ $(l \in \{1, ..., t\})$ then

$$v_{j_1}^1 \otimes v_{j_2}^2 \otimes ... \otimes v_{j_t}^t \in D.$$

Proposition 10. Let D be any subspace of $C = V^1 \otimes V^2 \otimes ... \otimes V^t$. Then there exists a normal subspace $\bar{D} \leq C$ with

$$(i) \quad \dim D = \dim \bar{D} \text{ and}$$
$$(ii) \quad | \operatorname{supp} D | \geq | \operatorname{supp} \bar{D} | .$$

Proof. We first apply Proposition 2 to D with respect to the codes $V = V_1$ and $W = V^2 \otimes ... \otimes V^t$. We obtain $D_1 \leq C$ with $\dim D = \dim D_1$ and $|\operatorname{supp} D| \geq |\operatorname{supp} D_1|$. Moreover D_1 is generated by elements of $\mathbf{V}^1 \otimes \mathbf{U}^2 \otimes ... \otimes \mathbf{U}^t = \{v^1_{i_1} \otimes u^2_{i_2} \otimes ... \otimes u^t_{i_t} \mid v^1_{i_1} \in \mathbf{V}^1, u^j_{i_j} \in \mathbf{U}^j\}$ for a suitable basis $\mathbf{U}_j = \{u^j_1, ..., u^j_{k_j}\}$ of V^j. For $2 \leq s \leq t$ we define recursively a space D_s in the following way. If D_{s-1} has already been constructed, we put $D_s = \bar{D}_{s-1}$ applying Lemma 8 with $V = V^s$ and $W = V^1 \otimes ... \otimes V^{s-1} \otimes V^{s+1} \otimes ... \otimes V^t$. Clearly, in each step we have $\dim D_s = \dim D_{s-1}$ and $|\operatorname{supp} D_s| \leq |\operatorname{supp} D_{s-1}|$. Finally let $\bar{D} = D_t$. It remains to prove that $\bar{D}$ is normal.

In order to see that we show by induction that for $s \in \{1, .., t\}$ the space D_s satisfies the following property.

Whenever
$$v^1_{i_1} \otimes ... \otimes v^s_{i_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s$$
and $0 \leq j_l \leq i_l$ ($l \in \{1, ..., s\}$) then
$$v^1_{j_1} \otimes ... \otimes v^s_{j_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s.$$

For $s = 1$ this is trivial. Assume that the property holds for $s - 1$, and that we have
$$v^1_{i_1} \otimes ... \otimes v^s_{i_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s$$
and $0 \leq j_l \leq i_l$ ($l \in \{1, ..., s\}$).

We first claim that $v^1_{j_1} \otimes ... \otimes v^{s-1}_{j_{s-1}} \otimes v^s_{i_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s$.

Note that for any $h \in \{1, ..., k_s\}$ such that
$$v^1_{i_1} \otimes ... \otimes v^{s-1}_{i_{s-1}} \otimes u^s_h \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_{s-1}$$
the inductive hypothesis implies
$$v^1_{j_1} \otimes ... \otimes v^{s-1}_{j_{s-1}} \otimes u^s_h \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_{s-1}.$$

Thus, if we construct D_s applying Lemma 8, we have
$$\dim D^{V^s}_{i_1,...,i_{s-1},i_{s+1},...,i_t} \leq \dim D^{V^s}_{j_1,...,j_{s-1},i_{s+1},...,i_t}.$$

This implies $v^1_{j_1} \otimes ... \otimes v^{s-1}_{j_{s-1}} \otimes v^s_{i_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s$, and the claim has been proved. Now, looking again at the definition of D_s, we obtain
$$v^1_{j_1} \otimes ... \otimes v^s_{j_s} \otimes u^{s+1}_{i_{s+1}} \otimes ... \otimes u^t_{i_t} \in D_s.$$

$\square$

Let t, r and k_i ($i \in \{1, ..., t\}$) be positive integers. We put

$$\mathcal{M}_t := \{\underline{i} = (i_1, ..., i_{t-1}) \mid 1 \leq i_j \leq k_j, 1 \leq j \leq t-1\}$$

for $1 < t$ and $\mathcal{M}_1 = \{0\}$. In the sequel we consider functions $\pi\colon \mathcal{M}_t \to \{0, 1, ..., k_t\}$ with $\pi(\underline{i}) = \alpha_{\underline{i}}$. We call π a $(k_1, k_2, ..., k_t)$-partition of r if

(i) $\Sigma_{\underline{i} \in \mathcal{M}_t} \pi(\underline{i}) = \Sigma_{\underline{i} \in \mathcal{M}_t} \alpha_{\underline{i}} = r$ and

(ii) π is a decreasing function in each coordinate, i. e.

$\alpha_{i_1,..,i_j,..,i_{t-1}} \leq \alpha_{i_1,..,i_j-1,..,i_{t-1}}$ for $j = 1, ..., t-1$ and $1 < i_j$.

If π is a $(k_1, k_2, ..., k_t)$-partition of r, we can associate to π a normal subspace of $V^1 \otimes ... \otimes V^t$ by setting

$$D = < v^1_{i_1} \otimes v^2_{i_2} \otimes ... \otimes v^t_{i_t} \mid i_t \leq \alpha_{i_1,...,i_{t-1}} > .$$

Note that $\dim D = r$. The fact that D is normal is a direct consequence of the definition of a $(k_1, k_2, ..., k_t)$-partition.
Conversely, if D is a normal subspace of $V^1 \otimes ... \otimes V^t$, we can define a $(k_1, k_2, ..., k_t)$-partition of $r = \dim D$ by putting

$$\alpha_{i_1,...,i_{t-1}} = \mid \{v^t_{i_t} \mid v^1_{i_1} \otimes ... \otimes v^t_{i_t} \in D\} \mid .$$

Proposition 11. Let D be a normal subspace of $V^1 \otimes ... \otimes V^t$ and let π be the associated $(k_1, k_2, ..., k_t)$-partition of $r = \dim D$. Then

$$\mid \mathrm{supp}\, D \mid = \Sigma_{\underline{i} \in \mathcal{M}_t} \Pi_{j=1}^{t-1} (d_{i_j}(V^j) - d_{i_j-1}(V^j)) d_{\alpha_{\underline{i}}}(V^t).$$

Proof. We argue by induction on t. For $t = 1$, this is trivial. Assume that the assertion holds true for a product of $t-1$ codes and consider

$$\mathcal{M}_{t-1} = \{\underline{\underline{i}} = (i_2, ..., i_{t-1}) \mid 1 \leq i_j \leq k_j, 2 \leq j \leq t-1\}.$$

For each $s \in \{1, ..., k_1\}$ let $r_s = \Sigma_{\underline{\underline{i}} \in \mathcal{M}_{t-1}} \alpha_{s,i_2,...,i_{t-1}}$ and

$$\begin{aligned} D_s &= < v^2_{i_2} \otimes ... \otimes v^t_{i_t} \mid i_t \leq \alpha_{s,i_2,...,i_{t-1}} > = \\ &= < v^2_{i_2} \otimes ... \otimes v^t_{i_t} \mid v^1_s \otimes v^2_{i_2} \otimes ... \otimes v^t_{i_t} \in D > \leq V^2 \otimes ... \otimes V^t. \end{aligned}$$

Obviously, D_s is normal and therefore

$$| \operatorname{supp} D_s |= \Sigma_{\underline{\underline{i}} \in \mathcal{M}_{t-1}} \Pi_{j=2}^{t-1} (d_{i_j}(V^j) - d_{i_j-1}(V^j)) d_{\alpha_{s,\underline{\underline{i}}}}(V^t)$$

by the inductive hypothesis. So we only have to prove that

$$| \operatorname{supp} D |= \Sigma_{s=1}^{k_1} (d_s(V^1) - d_{s-1}(V^1)) \mid \operatorname{supp} D_s \mid .$$

But $D_1 \geq D_2 \geq ... \geq D_{k_1}$ since D is normal. Therefore, the assertion is a direct consequence of the fact that V^1 satisfies the chain condition and $\{v_1^1, ..., v_{k_1}^1\}$ is a basis for the chain. $\square$

For a $(k_1, ..., k_t)$-partition of r let

$$\nabla(\pi) = \Sigma_{\underline{i} \in \mathcal{M}_t} \Pi_{j=1}^{t-1} (d_{i_j}(V^j) - d_{i_j-1}(V^j)) d_{\alpha_{\underline{i}}}(V^t).$$

As a consequence of Proposition 10 and Proposition 11 we get

Theorem 12. If each of the codes V^1,...,V^t satisfies the chain condition, then

$$d_r(V^1 \otimes ... \otimes V^t) = \min \{\nabla(\pi) \mid \pi \text{ a } (k_1, ..., k_t)\text{-partition of } r\}.$$

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