Cartan invariants and defect zero elements

R. Gow

Department of Mathematics
University College
Belfield, Dublin 4
Ireland

W. Willems

Institut für Algebra und Geometrie
Fakultät für Mathematik
Otto-von-Guericke-Universität
39016 Magdeburg
Germany

Abstract. In this note we prove relations between Cartan invariants in characteristc p and p-defect zero elements. Exploiting duality arguments in characteristic two we show that the existence of real 2-defect zero elements is equivalent to the existence of odd diagonal entries in the Cartan matrix corresponding to real-valued Brauer characters. In addition, we present an independent and elementary proof of Theorem 3.3 part (iii) in [2].

1 Introduction

Let G be a finite group and let K be a splitting field for KG of characteristic $p = 2$. In ([2], Theorem 3.3) the following surprising fact has been proved. There exists an odd Cartan invariant c_{ii} if and only if G contains a real element of 2-defect zero. By Brauer's permutation lemma the number of real 2-regular conjugacy classes of G is equal to the number of irreducible real-valued Brauer characters. Thus, for a group of even order, the condition c_{ii} odd implies the existence of a non-trivial irreducible real-valued Brauer character. In this short note we clarify the relation between irreducible real-valued Brauer characters and odd Cartan invariants. In particular, we obtain an independent and elementary proof of part (iii) of Theorem 3.3 in [2]. More generally, if K is of arbitrary characteristic p, we deduce a connection between the p-parts of the Cartan invariants and the defects of centralizers of p-regular elements.

Throughout this note we fix the following notations. Let K be a p-adic field and let R be the ring of integers in K with maximal ideal $\wp = \pi R$. Furthermore let $F = R/\wp$ denote the residue field of characteristic p. We suppose that K and F are splitting fields for the finite group G. By $\varphi_1, \ldots, \varphi_l$ we denote a full set of irreducible p-Brauer characters with corresponding projective characters $\Phi_1, \ldots, \Phi_l$. The character φ_1 stands for the trivial character. If $i \in \{1, \ldots, l\}$ we write $i = \bar{i}$ if φ_i is real-valued, or in other words, if the module affording the character φ_i is self-dual. Furthermore, let $g_1, \ldots, g_l$ denote a set of representatives for all p-regular classes in G. If ν_p denotes the p-valuation on $\mathbb{Z}$, i.e., $\nu_p(z) = n$ if p^n is the largest p-power dividing $z \in \mathbb{Z}$, then $\nu_p(|C_G(x)|)$ is usually called the p-defect of $x \in G$. Finally, let $C = (c_{ij})$ denote the Cartan matrix of FG.

2 Cartan invariants and centralizers of p-regular elements

In this section we assume that p is an arbitrary prime.

Theorem 2.1 *We always have*

$$\min_{i,j=1,\ldots,l} \nu_p(c_{ij}) = \min_{i=1,\ldots,l} \nu_p(|C_G(g_i)|).$$

Proof. Let $c = \min_{i,j=1,\ldots,l} \nu_p(c_{ij})$ and $d = \min_{i=1,\ldots,l} \nu_p(|C_G(g_i)|)$. By ([3], Chap. IV, 3.7), we know that

$$c_{ij} = \sum_{k=1}^{l} \Phi_i(g_k) \frac{\Phi_j(g_k^{-1})}{|C_G(g_k)|}.$$

Furthermore, by ([5], Lemma 2.21), we have $\frac{\Phi_i(g_k)}{|C_G(g_k)|} \in R$. This implies that

$$\nu_p(c_{ij}) \geq \min_{k=1,\ldots,l} \nu_p(\Phi_i(g_k)) \geq \nu_p(|C_G(g_k)|) \geq d$$

for all k. Thus $d \leq c$.

By ([3], Chap. IV, 3.8), we also have

$$\sum_{i,j=1}^{l} c_{ij} \varphi_j(g_k) \varphi_i(g_k^{-1}) = \sum_{i=1}^{l} \Phi_i(g_k) \varphi_i(g_k^{-1}) = |C_G(g_k)|$$

for all k. Thus $c = \min_{i,j=1,\ldots,l} \nu_p(c_{ij}) \leq \nu_p(|C_G(g_k)|)$ for all k. Hence $c \leq d$.

$\square$

Christine Bessenrodt noticed that a consideration of elementary divisors for the Cartan matrix also leads to a short proof of Theorem 2.1.

Corollary 2.2 *The following assertions are equivalent.*

a) There exists a Cartan invariant c_{ij} with $p \nmid c_{ij}$.

b) There exists an element $x \in G$ of p-defect zero.

Proof. This follows immediately from 2.1. $\square$

The last result may also be formulated in the following way. The prime p divides all Cartan invariants if and only if all elements of G are of positive p-defect.

Remarks 2.3 The condition b) in 2.2 does not imply the existence of a p-block of defect zero in general. As an example we may take the alternating group A_7 and $p = 2$. The Cartan matrix (see [4]) is given by

$$\begin{pmatrix} 4 & 2 & 3 & & & \\ 2 & 3 & 1 & & & \\ 3 & 1 & 2 & & & \\ & & & 2 & 1 & 2 \\ & & & 1 & 2 & 2 \\ & & & 2 & 2 & 4 \end{pmatrix}.$$

Note that there are four classes of 2-defect zero and four Cartan invariants c_{ij} with $j \leq i$ and c_{ij} odd. This coincidence also fails to be true in general as $G = \mathrm{A}_4$ already shows for $p = 2$. The Cartan matrix is given by

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

There are only two classes of 2-defect zero where both are not real.

3 Cartan invariants and 2-defect zero elements

Throughout this section we always assume that $p = 2$. In this situation duality arguments have turned out to be very powerful.

In [2] it has been proved that the condition c_{ii} odd for some i (not necessarily $i = \bar{i}$) is equivalent to the existence of a real 2-defect zero element. The proof relies on subtle computations in the group algebra, and is obtained as a by-product of more general results. As a consequence of Theorem 2.1 we get the following sharpening.

Theorem 3.1 *The following three conditions are equivalent.*

a) There exists an odd Cartan invariant c_{ij} where $i = \bar{i}$ and $j = \bar{j}$.

b) There exists a real element $x \in G$ of 2-defect zero.

c) There exists an odd Cartan invariant c_{ii} where $i = \bar{i}$.

Proof. $a) \Rightarrow b)$ As in Theorem 2.1 we have

$$c_{ij} = \sum_{k=1}^{l} \Phi_i(g_k) \frac{\Phi_j(g_k^{-1})}{|C_G(g_k)|}.$$

Now, since Φ_i and Φ_j are real-valued,

$$c_{ij} \equiv \sum_{k} \Phi_i(g_k) \frac{\Phi_j(g_k^{-1})}{|C_G(g_k)|} \mod \wp$$

where the sum runs only over real elements g_k. The pairing of g_k and g_k^{-1} if they are in different conjugacy classes, applied in conjunction with the argument used in the first part of the proof of Theorem 2.1, shows the existence of a real element g_k of 2-defect zero.
$b) \Rightarrow c)$ We use again the fact that

$$\sum_{i=1}^{l} \Phi_i(g_k)\varphi_i(g_k^{-1}) = |C_G(g_k)|$$

for all $k = 1, \ldots, l$. Now, if g_k is real, then

$$|C_G(g_k)| \equiv \sum\nolimits_{i=\bar{i}} \Phi_i(g_k)\varphi_i(g_k^{-1}) \mod \wp.$$

Furthermore for $i = \bar{i}$ we have

$$\Phi_i(g_k) \equiv \sum_{j=\bar{j}} c_{ij}\varphi_j(g_k) \mod \wp.$$

Thus

$$\begin{aligned} |C_G(g_k)| &\equiv \sum\nolimits_{i=\bar{i}} \Phi_i(g_k)\varphi_i(g_k) \mod \wp \\ &\equiv \sum\nolimits_{i=\bar{i}} \sum\nolimits_{j=\bar{j}} c_{ij}\varphi_j(g_k)\varphi_i(g_k) \mod \wp \\ &\equiv \sum\nolimits_{i=\bar{i}} c_{ii}\varphi_i(g_k)^2 \mod \wp. \end{aligned}$$

Since $2 \nmid |C_G(g_k)|$ there exists an index $i = \bar{i}$ with c_{ii} odd.
$c) \Rightarrow a)$ There is nothing to prove. □

Combining Theorem 3.1 and part (iii) of Theorem 3.3 in [2] we immediately obtain the next result. Here we give an independent and elementary proof which was brought to our attention by an anonymous referee of this note.

Theorem 3.2 *There exists an odd Cartan invariant c_{ii} if and only if there exists an odd Cartan invariant c_{ii} with $i = \bar{i}$.*

Proof. We claim that if c_{ij} is odd and $\Phi_i \Phi_{\bar{j}}$ is real then there is a real irreducible constituent of $\Phi_i \Phi_{\bar{j}}$ with index $k = \bar{k}$ such that c_{kk} is odd. The nontrivial direction of Theorem 3.2 then follows by putting $j = i$.

In order to prove the claim first note that $\Phi_i \Phi_{\bar{j}} = \Phi_{\bar{i}} \Phi_j$. Furthermore, for a 2-regular element $g \in G$ we have

$$\chi(g)^2 \equiv \chi(g^2) \bmod \wp \quad \text{and} \quad |C_G(g)| = |C_G(g^2)|.$$

Finally, $g \mapsto g^2$ defines a bijection on the 2-regular elements. If $[\cdot,\cdot]$ denotes the usual scalar product on the ring of class functions we therefore obtain

$$\begin{aligned} [\Phi_i \Phi_{\bar{j}}, \Phi_i \Phi_{\bar{j}}] &= \textstyle\sum_{k=1}^{l} \frac{\Phi_i(g_k)^2 \Phi_{\bar{j}}(g_k)^2}{|C_G(g_k)|} \\ &\equiv \textstyle\sum_{k=1}^{l} \frac{\Phi_i(g_k^2) \Phi_{\bar{j}}(g_k^2)}{|C_G(g_k^2)|} \bmod \wp \\ &\equiv \textstyle\sum_{k=1}^{l} \frac{\Phi_i(g_k) \Phi_{\bar{j}}(g_k)}{|C_G(g_k)|} = [\Phi_i, \Phi_j] = c_{ij} \bmod \wp. \end{aligned}$$

Now write $\Phi_i \Phi_{\bar{j}} = \sum_{k=1}^{l} a_k \Phi_k$ with nonnegative integers a_k. Thus

$$\begin{aligned} [\Phi_i \Phi_{\bar{j}}, \Phi_i \Phi_{\bar{j}}] &= \textstyle\sum_{k,k'=1}^{l} a_k \, a_{k'} [\Phi_k, \Phi_{k'}] \\ &\equiv \textstyle\sum_{k=1}^{l} a_k^2 [\Phi_k, \Phi_k] \bmod \wp \\ &\equiv \textstyle\sum_{k=\bar{k}}^{l} a_k^2 [\Phi_k, \Phi_k] \bmod \wp \end{aligned}$$

where the last equality follows from the fact that $a_k = a_{\bar{k}}$ and $[\Phi_k, \Phi_k] = [\Phi_{\bar{k}}, \Phi_{\bar{k}}]$ for all k. The condition c_{ij} odd therefore implies the existence of a $k = \bar{k}$ with $[\Phi_k, \Phi_k] = c_{kk}$ odd. □

Remark 3.3 Let $G = \mathrm{A}_5$ and $p = 2$. Thus the Cartan matrix is of shape

$$\begin{pmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since all Brauer characters are real-valued the example shows that there is no relation between the number of odd Cartan invariants c_{ii} with $i = \bar{i}$ and the number of real classes of 2-defect zero. Note that the latter number is 3. Moreover the example proves that in general the equivalence of a) and c) in Theorem 3.1 does not allow a block version.

Remark 3.4 Let $G = \Gamma(2^3)$ be the affine group of order 168 and let $p = 3$. Since G is 3-nilpotent the Cartan matrix is diagonal, and one easily computes that

$$C = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The two blocks of defect zero are dual to each other and the other two are self-dual. Thus $p \nmid c_{ii}$ for some i does not imply $p \nmid c_{ii}$ for a suitable $i = \bar{i}$. Furthermore there is no real element of 3-defect zero.

In characteristic two a nontrivial irreducible self-dual module M is always of symplectic type, i.e. M always carries a non-degenerate G-invariant symplectic form (Fong's Lemma). It may or may not have a non-degenerate G-invariant quadratic form from which the symplectic form derives via polarization (see [1], Chap. VII, section 8). We say that M is of $+$ type if there is a G-invariant quadratic form defined on the module, and that M is of $-$ type if there is no G-invariant quadratic form defined on the module.

According to Theorem 3.1 we ask the following natural questions. If c_{ij} is odd for $i = \bar{i}$ and $j = \bar{j}$ is there a connection between the geometries of the modules corresponding to i and j. Moreover, is there always an i with $i = \bar{i}$ and corresponding module of $+$ type? Unfortunately both questions have a negative answer.

Examples 3.5 For necessary details the reader is referred to [4]. Cartan matrices are easy to compute from the decomposition matrices which are available in [6]. Let $p = 2$.
a) If $G = L_2(8)$ then all G-modules in characteristc two are self-dual, and there are irreducible modules of dimensions 2 and 4, respectively, with indices 2 and 7 such that $1 = c_{2,7}$. The irreducible module of dimension 2 is geometrically of $-$ type whereas the irreducible module of dimension 4 is of $+$ type. Thus c_{ij} odd does not relate the geometry of self-dual modules associated to i and j.
b) If G is the Mathieu group M_{11} then the diagonal entries of the Cartan matrix are $4, 5, 3, 1, 1$ where the last two modules are not self-dual. The second and third modules are of $+$ type. If G is the Mathieu group M_{22} then the diagonal entries of C are $92, 21, 21, 23, 3, 3, 6$. The modules corresponding to the indices $2, 3, 5, 6$ are not self-dual and the module corresponding to 4 is of $-$ type. Thus c_{ii} odd does not give any information about the geometry of the module associated to i.

Proposition 3.6 *If $|G|$ is even then c_{11} is even.*

Proof. First note that nontrivial real irreducible 2-modular Brauer characters have even degree (see for instance, [5], Theorem 2.30). Putting $g_k = 1$ we obtain directly from the proof b) $\Rightarrow$ c) of Theorem 3.1

$$|G| = |C_G(g_k)| \equiv \sum_{i=\bar{i}} c_{ii}\varphi_i(g_k)^2 \equiv c_{11} \mod 2.$$

□

For p odd the analogue of 3.6 is far from being true, as the symmetric group S_3 for $p = 3$ already shows.

Proposition 3.7 *If all diagonal entries c_{kk} in the Cartan matrix are even and for some $i = \bar{i}$ all c_{ij} are odd for $j \neq i$ then G has a normal Sylow 2-subgroup T with $C_G(T) \leq T$.*

Proof. Let c_{ij} be odd for all $j \neq i = \bar{i}$. If $j = \bar{j} \neq i$ then by Theorem 3.1 we obtain c_{kk} odd for some k, a contradiction to the assumptions in the Proposition. In particular, $i = 1$, since otherwise we may choose $j = 1$. Thus there is no $j \neq 1$ with $j = \bar{j}$. This shows that the trivial module is the only irreducible self-dual module. A standard argument using Brauer's permutation lemma and a theorem of Baer (see [5], 2.33) shows that the Sylow 2-subgroup is normal in G. If $C_G(T) \not\subseteq T$ then $C_G(T) = T_0 \times N$ where T_0 is a Sylow 2-subgroup of $C_G(T)$ and $1 \neq N$ is a normal $2'$-subgroup of G. This means that G has at least two blocks contradicting the assumption that c_{ij} is odd for all $j \neq i$. □

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