Brauer character degrees and class lengths of p-power order*

Wolfgang Willems

Abstract Let G be a finite group. We prove: If p is an odd prime and all irreducible p-Brauer characters have degrees which are powers of p then G is solvable and the Sylow subgroups of a p-complement are metabelian. As an analogue, for class lengths we have. If p is any prime then all p-regular conjugacy classes have p-power lengths if and only if G is solvable with an abelian p-complement.

Mathematics Subject Classification (2000): 20C20
Keywords: Brauer characters, character degrees, solvable groups, simple groups.

1 Introduction and results

It is well known that the set of degrees of irreducible p-Brauer characters of a finite group G reflect somehow the structure of G. The same holds true if we replace the degrees by the lengths of p'-conjugacy classes. For instance, if the degrees of all irreducible p-Brauer characters are prime to p then, by a result of Ito-Michler [18], G has a normal Sylow p-subgroup. According to a result of Camina [3], the condition that the lengths of all p'-conjugacy classes are prime to p is equivalent to $G = O_p(G) \times O_{p'}(G)$. In this note we are looking at the other extrem, i.e. that the degrees of irreducible p-Brauer characters or the lengths of p-regular conjugacy classes are powers of p.

We prove

Theorem 1.1 *Let G be a finite group and let p be an odd prime. If all irreducible p-Brauer characters have p-power degrees then G is solvable and all Sylow subgroups of a p-complement are metabelian.*

For $p = 2$ this is far away to be true and we conjecture here

*Supported by the Sino-German Center for Research Promotion.

Conjecture 1.2 *Let G be a finite non-abelian simple group. If all irreducible 2-Brauer characters have 2-power degree then G is isomorphic to one of the following groups:*

$$\mathrm{SL}(2,2^f),\ \mathrm{Suz}(2^{2f+1}),\ \mathrm{Sp}(4,2^f)\ \textit{or}\ \mathrm{PSL}(2,9).$$

In order to prove Theorem 1.1 we only have to show that G is solvable. The statement that the Sylow subgroups of a p-complement are metabelian follows directly from an early work of Manz and Wolf [17]. To prove solvability we may assume that $p \mid |G|$. For, if $p \nmid |G|$, then the p-Brauer characters are classical characters. Thus all degrees are 1 since they devide the order of G. This forces G to be abelian and we are done.

Reduction step to simple groups. Suppose that each simple non-abelian group G with $p \mid |G|$ has an irreducible p-Brauer character whose degree is not a power of p. Under this assumption Theorem 1.1 follows by the following argument.

If G is not solvable let

$$X = X_1 \times \ldots \times X_s$$

be a chief factor of G with X_i non-abelian, simple and $p \mid |X|$. If X_i has an irreducible p-Brauer character whose degree is not a power of p then the same holds true for X, and by Clifford's Theorem, G has an irreducible Brauer character whose degree is not a power of p.

Thus it remains to prove

Proposition 1.3 *Let G be a finite non-abelian simple group with $2 \neq p \mid |G|$. Then G has an irreducible p-Brauer character whose degree is not a power of p.*

Analoguously to Theorem 1.1, for conjugacy classes we have

Theorem 1.4 *Let p be a prime. The following statements are equivalent.*

a) $|G : C_G(x)|$ is a power of p for all p'-elements $x \in G$.

b) G is solvable with an abelian Hall p'-subgroup.

Proof: The implication b) to a) is obvious. Thus we may assume that condition a) is given. In order to show that G is solvable we argue by induction on $|G|$. By a result of Burnside (see [15], Kap. V, Satz 7.2), the group G is not simple since there is a class length of p-power order. Let $1 < N \lhd G$. Since the lengths of the conjugacy classes of N and G/N are divisors of the lengths of conjugacy classes of G, both N and G/N are solvable by induction. Thus G is solvable.

Now let H be a p-complement in G and let N be a minimal normal subgroup of G. Again by induction on $|G|$, we may assume that N is a p'-group and that N is the only minimal normal subgroup. Since G is solvable, N is a q-group for some prime $q \neq p$. Thus we have

$$N \leq O_q(G) = \mathrm{Fit}(G).$$

By the assumption in a), $C_G(x) \geq \mathrm{Fit}(G)$ for all p-regular elements x. Since for solvable groups the centralizer of the Fitting group is contained in the Fitting group we get

$$\mathrm{Fit}(G) \geq C_G(\mathrm{Fit}(G)) \geq \langle x \mid x \ p\text{-regular}\rangle.$$

In particular, $\mathrm{Fit}(G)$ is the Hall p'-subgroup and the assumption that all p'-conjugacy classes have p-power size forces $\mathrm{Fit}(G)$ to be abelian. $\square$

Corollary 1.5 *Let p be a prime and suppose that the lengths of all p'-conjugacy classes are powers of p. Then the degrees of all irreducible p-Brauer characters are also powers of p.*

Proof: By Theorem 1.4, the group G is solvable with abelian p-complement H. Let Φ denote the projective character corresponding to an irreducible p-Brauer character φ. By a result of Fong (see [11], Chap. X, 3.2), we have

$$\Phi(1) = |G|_p \varphi(1)_{p'}$$

where $|G|_p$ denotes the p-part of the order of G and $\varphi(1)_{p'}$ the p'-part of the degree of φ. By a further result, also due to Fong (see [11], Chap. X, 3.4), Φ is induced from an irreducible so-called Fong character χ of a p-complement of G. Since H is abelian $\chi(1) = 1$, hence $\Phi(1) = |G|_p$. This shows that $\varphi(1)_{p'} = 1$. Thus all irreducible Brauer characters have p-power degrees. $\square$

The following example may illustrate the difference between Theorem 1.1 and Theorem 1.4.

Example 1.6 Let $G = (C_3 \times C_3)\mathrm{SL}(2,3)$ where $\mathrm{SL}(2,3)$ acts faithfully on the normal subgroup $C_3 \times C_3$ and let $p = 2$. One easily checks that all irreducible 2-Brauer characters have 2-power degrees and that there is a 3-conjugacy class whose size is not a power of 2. Clearly, the Sylow 3-subgroups are not abelian which is in accordance with Theorem 1.4.

The next sections are devoted to prove Proposition 1.3 from which the main Theorem 1.1 follows as shown above. We distingish between the different types of non-abelian simple groups. Thus the proof for character degrees depends on the classification of finite simple groups whereas the proof for conjugacy classes does not.

2 Sporadic groups

Proposition 2.1 *Let G be a finite simple sporadic group and let p be an arbitrary prime. Then G has an irreducible p-Brauer character whose degree is not a power of p.*

Proof. To find an irreducible p-Brauer character whose degree is not a power of p we preceed as follows. For $p = 2$, note that the first Conway group Co_1 has an irreducible representation of degree 24 and that the Baby Monster B has an irreducible representation of degree 4370 (see [22]). In all other cases we find an irreducible p-Brauer character whose degree is not a power of p either by checking directly the known list of irreducible p-Brauer degrees as given in the Aachen library [23] or by checking the degrees of classical irreducible characters of defect zero via the Atlas of Finite Groups [7]. □

3 Alternating groups

In 2006, Christine Bessenrodt [1] studied irreducible p-Brauer characters for symmetric and alternating groups which have p-power degrees. For alternating groups her result answers both Proposition 1.3 and the Conjecture 1.2 as well. More precisely, she proved

Proposition 3.1 *Let $p \leq n$ be a prime.*

All irreducible p-Brauer characters of the alternating group A_n are of p-power degree if and only if $p = 2$ and $n \leq 6$, or $p = 3$ and $n \leq 4$.

Thus only for $p = 2$ there are simple alternating groups for which all irreducible p-Brauer characters have p-power degrees, namely A_5 and A_6. This is in accordance with Conjecture 1.2 since $\mathrm{A}_5 = \mathrm{SL}(2,4)$ and $\mathrm{A}_6 = \mathrm{PSL}(2,9)$.

4 Groups of Lie type in defining characteristic $r = p \neq 2$

In this section we consider finite simple groups L of Lie type defined over fields of charactersitic p and its p-Brauer characters. Since $p = r$ is odd such a simple group is a Chevalley group or a twisted version unless $L = {}^2G_2(3)$ (see [4], section 11.1 and 14.4). In the exceptional case we have $L' = {}^2G_2(3)' = A_1(8)$ which is a group we do not have to consider here since $r \neq 2$. For the group L let $\mathcal{L}$ denote the corresponding Lie algebra defined over an algebraically closed field $\bar{K}$ of characteristic p containing K and let $\mathrm{Z}(\mathcal{L})$ denote its center. Then $\mathcal{L}$ is a $\bar{K}$L-module, the so-called adjoint module. Since a twisted Chevalley group is a subgroup of a Chevalley group the space L is also a module for twisted type groups.

In the following table the dimension of $\mathcal{L}$ is taken from ([4], 3.6) and the dimension of $Z(\mathcal{L})$ from [14].

Type	dim $\mathcal{L}$	dim $\mathrm{Z}(\mathcal{L})$
A_l $(l \geq 1)$	$l(l+2)$	0, or 1 if $p \mid l+1$
B_l $(l \geq 2)$	$l(2l+1)$	0
C_l $(l \geq 3)$	$l(2l+1)$	0
D_l $(l \geq 4)$	$l(2l-1)$	0
G_2	14	0
F_4	52	0
E_6	76	0, or 1 if $p = 3$
E_7	133	0
E_8	248	0

Since p is odd, by a result of Hiss [12], the module $\mathcal{L}/\mathrm{Z}(\mathcal{L})$ is always irreducible, for a twisted and a non-twisted Chevalley group L, unless L is of type G_2 and $p = 3$. In the exceptional case $\mathcal{L} = \mathcal{L}/\mathrm{Z}(\mathcal{L})$ has two composition factors both of dimension 7 which is not a power of 3. In all other cases (in which $\mathcal{L}/Z(\mathcal{L})$ is irreducible) one easily checks using the table above that $\dim \mathcal{L}/\mathrm{Z}(\mathcal{L})$ is not a power of p except L is of type A_1 and defined over the prime field $\mathbb{F}_3$. But this group is solvable of order 12 and we do not have to consider.

Thus we have shown

Proposition 4.1 *Let p be an odd prime. If L is a finite simple group of Lie type defined over a field of characteristic p then G has an irreducible p-Brauer character whose degree is not a power of p.*

5 Groups of Lie type in non-defining characteristic $r \neq p$

Let L be a finite simple non-abelian group of Lie type defined over a finite field $\mathbb{F}_q$ where q is a power of a prime $r \neq p \neq 2$. Such a group can be obtained as follows (see [5]). There exists a simply connected simple linear algebraic group G and an endomorphism, say σ, of G such that $L = G_\sigma/\mathrm{Z}(G_\sigma)$ unless L is the Tits simple group which is the commutator group of ${}^2F_4(2)$. Here G_σ denotes the subgroup of fixed points of σ acting on G. According to [7] the Tits simple group ${}^2F_4(2)'$ has for all primes p an irreducible complex character of p-defect zero whose degree is not a power of p. Thus we may assume that

$$L = G_\sigma/\mathrm{Z}(G_\sigma).$$

In order to show that L always has an irreducible p-Brauer character whose degree is not a power of p we proceed as follows. Either we find such a character in a known list of p-Brauer characters or we search for an irreducible complex character χ of p-defect zero, i.e. $|L|_p \mid \chi(1)$.

First we are looking at groups of small rank, more precisely at the groups

$$(*) \quad A_l(q)\,(l = 1, 2), {}^2A_2(q), {}^2B_2(2^{2m+1}), {}^2G_2(3^{2m+1}), {}^3D_4(q) \text{ and } G_2(q).$$

Lemma 5.1 *For p odd, each of the groups listed in $(*)$ have an irreducible p-Brauer character whose degree is not a power of p*

Proof: For $l = 1$ the groups $A_l(q)$ may be checked with [2]. According to [19] the groups $A_2(q)$ and ${}^2A_2(q)$ have irreducible p-Brauer characters which are not of p-power degree. For the Suzuki groups ${}^2B_2(2^{2m+1})$ the assertion follows from the character table given in ([16], Chap. XI, 5.10). Using Hiss' results [13] we see that the same holds true for the Ree groups ${}^2G_2(3^{2m+1})$. For the Steinberg triality ${}^3D_4(q)$ Deriziotis and Michler [8] computed the character table and the block decomposition from which the assertion also follows. Thus we are left with the groups $G_2(q)$. For $2, 3 \nmid q$, Chang and Ree computed the character table in [6]. For $q = 3^f$, Enomoto constructed the irreducible characters in [9] and finally, for q even, this has been done by Enomoto and Yamada in [10]. For all primes $p \neq 2$ one easily finds irreducible complex characters of p-defect zero whose degrees are not powers of p.

Let $\mathcal{S}$ be the set of finite simple groups of Lie type defined over a field of characteristic $r \neq p$ which are not isomorphic to groups in $(*)$.

Proposition 5.2 *If p is odd and G is a finite simple group of Lie type belonging to $\mathcal{S}$ then G has an irreducible p-Brauer character whose degree is not a power of p.*

Proof: Using Deligne-Lusztig characters Michler proved in [18] that for p odd and any finite simple group of Lie type defined in characteristic $r \neq p$ there exists an irreducible classical character, say χ of p-defect zero. Moreover, such a χ has degree

$$\chi(1) = |L : T|_{r'}$$

where T is the fixed point group under σ of a maximal σ-stable torus in G.

Suppose that

$$\chi(1) = |L : T|_{r'} = p^s$$

for some s.

Going through the types of groups in $\mathcal{S}$ and checking the divisors of degrees of p-defect zero characters χ found in [18] we get

groups	divisors of $\chi(1)$	exceptions
$A_l(q)\,(l \geq 3)$	$(q^2-1)(q^2+1)$ $(q^2-1)(q^3-1)$	
${}^2A_l(q)\,(l \geq 3)$	$(q^2-1)(q^2+1)$ $(q^2-1)(q^3+1)$	
$B_2(q)$	(q^2-1)	$B_2(2)$
$B_l(q)$, $C_l(q)\,(l \geq 3)$	$(q^2-1)(q^2+1)$	
$D_l(q)$, ${}^2D_l(q)\,(l \geq 4)$	$(q^2-1)(q^2+1)$ $(q^2-1)(q^6-1)$	$D_4(2)$, $p=3$
$F_4(q)$, ${}^2F_4(q)$	$(q^2-1)(q^2+1)$	${}^2F_4(2)$
$E_6(q), {}^2E_6(q)$	$(q^2-1)(q^2+1)$	
$E_7(q)$	$(q^2-1)(q^2+1)$	$p \mid (q+1)$
$E_8(q)$	$(q^2-1)(q^2+1)$	

Note that all divisors in the second column are not powers of $p \neq 2$. Thus it remains to deal with the exceptions. First observe that $B_2(2)' \cong \mathrm{A}_6$ which we already checked in Proposition 3.1. For $p = 3$ the group $D_4(2) = O_8^+(2)$ has an irreducible 3-Brauer character of degree $4 \cdot 3^5$ (see [7]). The Tits simple group ${}^2F_4(2)'$ we checked earlier. Finally we have to deal with $E_7(q)$ for odd primes $p \mid q+1$. For all such primes p, by ([18], p. 225), there exists an irreducible complex character Λ of p-defect zero with

$$\Lambda(1) = \frac{1}{2} q \phi_2^7 \phi_4^2 \phi_6^3 \phi_8 \phi_{10} \phi_{12} \phi_{14} \phi_{18}$$

where ϕ_k denotes the kth cyclotomic polynomial in q. If $\Lambda(1)$ were a power of p then $p \mid \phi_2 = q+1$ and $p \mid \phi_4 = q^2+1$, hence $p \mid \gcd(q+1, q^2+1) \in \{1, 2\}$, a contradiction to the assumption that p is odd. $\square$

Section 4 and 5 show that a finite simple group of Lie type always has an irreducible p-Brauer character of degree which is not a power of p if p is odd. This completes the proof of Proposition 1.3, hence of Theorem 1.1.

Remarks 5.3 a) We consider again the equation

$$(*) \qquad \chi(1) = |L : T|_{r'}$$

stated in the proof of Proposition 5.2. If the defining characteristic r is odd then the Sylow 2-subgroup of the underlying simple Lie group L must be abelian by $(*)$. Thus, by a result of Walter [20], L is isomorphic to $\mathrm{PSL}(2,q)$ with $q \equiv 3,5 \bmod 8$ or to a Ree group ${}^2G_2(3^{2m+1})$. But both groups do not belong to the set of groups belonging to $\mathcal{S}$. Hence in Proposition 5.2 one only has to check finite simple groups of Lie type in defining characteristic 2.
b) A confirmation of Conjecture 1.2 should be accessible along the same lines as in this note replacing [18] by [21] in which the existence of 2-defect zero characters for all finite simple groups of Lie type has been proved.

References

[1] C. Bessenrodt. Irreducible p-Brauer characters of p-power degree for symmetric and alternating groups. Preprint 2006.

[2] R. Burkhardt. Die Zerlegungsmatrizen der Gruppen $\mathrm{PSL}(2,p^f)$. J. Algebra **40** (1976), 75-96.

[3] A.R. Camina. Arithmetical conditions on the conjugacy class numbers of a finite group. J. London Math. Soc **5** (1972), 127-132.

[4] R.W. Carter. *Simple groups of Lie type.* John Wiley, London 1972.

[5] R.W. Carter. *Finite groups of Lie type: Conjugacy classes and complex characters.* John Wiley, Chichester 1985.

[6] B. Chang and R. Ree. The characters of $G_2(q)$. Symposia Math. **13** (1974), 395-413.

[7] J.H. Conway at al.. *Atlas of finite groups*, Clarendon Press, Oxford 1985.

[8] D.I. Deriziotis and G.O. Michler. Character table and blocks of the finite triality groups ${}^3D_4(q)$. Trans. Am. Math. Soc. **303** (1987), 39-70.

[9] H. Enomoto. The characters of the finite Chevalley group $G_2(q)$, $q = 3^f$. Japan J. Math. **2** (1976), 191-248.

[10] H. Enomoto and H. Yamada. The characters of $G_2(2^n)$. Japan J. Math. **12** (1986), 325-377.

[11] W. Feit. *The representation theory of finite groups.* North Holland, Amsterdam 1982.

[12] G. Hiss. Die adjungierten Darstellungen der Chevalley-Gruppen. Archiv Math. **42** (1984), 408-416.

[13] G. Hiss. The Brauer trees of the Ree groups. Comm. Algebra **19** (1991), 871-888.

[14] J.F. Hurley. Centers of Chevalley algebras. J. Math. Soc. Japan **34** (1982), 219-222.

[15] B. Huppert. *Endliche Gruppen I.* Springer Verlag, Berlin, Heidelberg, New York, 1967.

[16] B. Huppert and N. Blackburn. *Finite Groups II, III.* Springer Verlag, Berlin, Heidelberg, New York, 1982.

[17] O. Manz and T. Wolf. Brauer characters of q'-degree in p-solvable groups. J Algebra **115** (1988), 75-91.

[18] G.O. Michler. A finite simple group of Lie type has p-blocks of different defects, $p \neq 2$. J. Algebra **104** (1986), 220-230.

[19] W.A. Simpson and J.S. Frame. The character tables of $\mathrm{SL}(3, q)$, $\mathrm{SU}(3, q^2)$, $PSL(3, q)$, $\mathrm{PSU}(3, q^2)$. Canad. J. Math. **25** (1979), 486-494.

[20] J.H. Walter. The characterization of finite groups with abelian Sylow 2-subgroups. Ann. of Math. **89** (1969), 405-514.

[21] W. Willems. Blocks of defect zero in finite simple groups. J. Algebra **113** (1988), 511-522.

[22] R.A. Wilson et al.. *Atlas of finite group representations - version 3* `http://brauer.maths.qmul.ac.uk/Atlas/v3`

[23] R.A. Wilson et al.. *The modular atlas homepage* `http://www.math.rwth-aachen.de/~MOC`

Wolfgang Willems
Otto-von-Guericke-Universität
Institut für Algebra und Geometrie
39016 Magdeburg

email:wolfgang.willems@mathematik.uni-magdeburg.de