QUASI-PROJECTIVE AND QUASI-LIFTABLE CHARACTERS

W. WILLEMS AND A.E. ZALESSKI

Abstract We study ordinary characters of a finite group G which vanish on the p-singular elements for a fixed prime p dividing the order of G. Such characters are called quasi-projective. We show that all quasi-projective characters of G are characters of projective modules if and only if the ordinary irreducible characters of G can be ordered in such a way that the top square fragment of the decomposition matrix is diagonal. Finally, we prove that the number of indecomposable quasi-projective characters of G is finite and characterize them in case of blocks with cyclic defect groups.

1. Introduction

Throughout this paper let p always be a prime and let G be a finite group. The order of G is denoted by $|G|$, and $|G|_p$ stands for the p-part of $|G|$. For $n, m \in \mathbb{N}$ the notation $n \mid m$ means that n divides m. By $\mathrm{Irr}(G)$ we denote the set of classical irreducible characters of G and by $\mathrm{IBr_p(G)}$ that of irreducible p-Brauer characters with respect to a splitting p-modular system. We write Φ_φ for the ordinary character associated to the projective cover of the module corresponding to $\varphi \in \mathrm{IBr_p(G)}$. If χ is an ordinary character of G then χ° denotes the restriction of χ on the set of p-regular elements. Furthermore $(\cdot\,,\cdot)$ stands for the usual scalar product on the complex vector space of ordinary characters and $(\cdot\,,\cdot)^\circ$ means its restriction to the set of p-regular elements.

Definition 1.1. *An ordinary character* Λ *of* G *is called quasi-projective if*

$$\Lambda = \sum_{\varphi \in \mathrm{IBr_p(G)}} a_\varphi \Phi_\varphi \text{ with } a_\varphi \in \mathbb{Z}.$$

In case $a_\varphi \geq 0$ *for all* $\varphi \in \mathrm{IBr_p(G)}$ *we call* Λ *projective, i.e.,* Λ *is the ordinary character of a projective module.*

Note that a character Λ is quasi-projective if and only if Λ vanishes on the set of p-singular elements of G (see [14, Theorem 2.13 and Corollary 2.17]). This shows that the coefficients a_φ are automatically integers since $a_\varphi = (\Lambda, \varphi)^\circ = (\Lambda, \varphi)$ and φ is an integer linear combination of ordinary irreducible characters restricted to p-regular elements (see [14, Corollary 2.16]). Although quasi-projective characters are a natural generalization of characters of projective modules, they did not attract much attention so far. In [15] the authors study quasi-projective characters of degree $|G|_p$, but only for Chevalley groups in defining characteristic p.

Definition 1.2. *We call a* p*-Brauer character* φ *quasi-liftable if there exists an ordinary character* χ *such that* $\chi^\circ = b\varphi$ *with* $b \in \mathbb{N}$.

Quasi-liftable irreducible Brauer characters φ for which $b > 1$ are of interest since they allow non-split self-extensions. More precisely, suppose that $\chi^\circ = b\varphi$ with $b > 1$. Let V be the module in characteristic p affording φ. According to a result of Thompson ([6, Ch. I, Theorem 17.12]) there is a lattice affording χ whose Brauer reduction mod p is indecomposable. This implies in particular that there is an indecomposable module which is an extension of V by V, i.e., $\mathrm{Ext}^1_G(V,V) \neq 0$. One says that V has a self-extension. Surprisingly, for most Chevalley groups in defining characteristic $p > 3$ even liftable modules have self-extensions [16, Proposition 1.4]. As shown in [2] the self-extension phenomenon is very rare.

Quasi-liftable irreducible Brauer characters which are not liftable are hard to find, but they exist. Take for instance $G = 2.\mathrm{M}_{12}.2$, where M_{12} is the Mathieu group, and $p = 2$. Then there exists a non-liftable Brauer character $\varphi \in \mathrm{IBr}_2(\mathrm{G})$ of degree 44 and a $\chi \in \mathrm{Irr}(G)$ of degree 88 such that $\chi^\circ = 2\varphi$ (see decomposition matrices in [13]). Another example is provided by $G = ON.2$ again for $p = 2$. In this case there exists a $\chi \in \mathrm{Irr}(G)$ of degree 51832 with $\chi^\circ = 2\varphi$ where $\varphi \in \mathrm{IBr}_2(\mathrm{G})$ is not liftable.

These examples may suggest that $p \mid b$ if $\chi^\circ = b\varphi$ for $\varphi \in \mathrm{IBr}_\mathrm{p}(\mathrm{G})$ and $b > 1$. However this is not true. The group $G = {}^2F_4(2)'.2$ contains a non-liftable $\varphi \in \mathrm{IBr}_2(\mathrm{G})$ of degree 26 and characters $\chi, \psi \in \mathrm{Irr}(G)$ with $\chi^\circ = 2\varphi$ and $\psi^\circ = 3\varphi$. The principal 2-block contains the trivial Brauer character (which is liftable), φ (which is quasi-liftable, but not liftable) and a non-quasi-liftable irreducible Brauer character of degree 246.

By the above examples one might be tempted to conjecture that the existence of quasi-liftable, but not liftable irreducible p-Brauer character occurs only for $p = 2$. This is not the case.

Example 1.3. Let $p = 3$ and let $H = 3.\mathrm{A}_6 \times \mathrm{E}_{27}$ where E_{27} is an elementary abelian group of order 27. The centers $Z(3.\mathrm{A}_6)$ and $Z(E_{27})$ are generated by elements of order 3, say z and z' respectively. The group $3.\mathrm{A}_6$ has an irreducible 3-Brauer character φ which is liftable, say to χ. For E_{27} we can choose an irreducible complex character ψ of degree 3 such that (z, z') is in the kernel of $\chi\psi \in \mathrm{Irr}(H)$. Thus $\chi\psi$ is an irreducible character of degree 27 of

$$G = 3.\mathrm{A}_6 \times \mathrm{E}_{27}/\langle(\mathrm{z},\mathrm{z}')\rangle \cong 3.\mathrm{A}_6 \curlyvee \mathrm{E}_{27}$$

which is a central product. Clearly $(\chi\psi)^\circ = 3\varphi$ and φ is not liftable.

If G is p-solvable then every quasi-projective character is the character of a projective module [4, Theorem 32.17]) or [14, Lemma 10.16]. It is well known that every irreducible p-Brauer character of a p-solvable group is liftable to an ordinary character. The following result which we prove in section 2 generalizes this.

Theorem 1.4. *Let G be a finite group and let p be a prime. Then the following are equivalent:*

a) *Every quasi-projective character $\Lambda = \sum_{\varphi \in \mathrm{IBr}_\mathrm{p}(\mathrm{G})} a_\varphi \Phi_\varphi$ of G is projective.*
b) *Every $\varphi \in \mathrm{IBr}_\mathrm{p}(\mathrm{G})$ is quasi-liftable.*

We would like to mention here that the statement in b) is equivalent to the fact that the ordinary irreducible characters of G can be ordered in such a way that the top square fragment of the decomposition matrix is diagonal. Furthermore, the proof of Theorem 1.4 together with [6, Ch. IV, Lemma 3.14] shows that there is a blockwise version of Theorem 1.4.

The groups for which all irreducible p-Brauer characters are liftable have been studied extensively by Hiss in [8] and [7]. Unfortunately, a full classification is still not available. So we can not expect a classification of all groups for which all irreducible p-Brauer characters are quasi-liftable, even if we assume that this holds true for all primes p.

For a fixed prime p the class of finite groups for which all irreducible p-Brauer characters are quasi-liftable may be bigger than that for which all irreducible p-Brauer characters are liftable. Take for instance $p = 2$. Then all irreducible 2-Brauer characters of $2.\mathrm{M}_{12}.2$ are quasi-liftable but not all are liftable.

However, in section 3 we show

Theorem 1.5. *If G is a finite quasi-simple group then the following are equivalent:*

a) *For all primes p every $\varphi \in \mathrm{IBr_p(G)}$ is quasi-liftable.*
b) $G = \mathrm{SL}(2, 5)$.
c) *For all primes p every $\varphi \in \mathrm{IBr_p(G)}$ is liftable.*

Thus one is naturally tempted to ask:

Question 1.6. Let G be an arbitrary finite group. Are the following equivalent?

a) For all primes p every $\varphi \in \mathrm{IBr_p(G)}$ is quasi-liftable.
b) For all primes p every $\varphi \in \mathrm{IBr_p(G)}$ is liftable.

In the last section we prove that for a finite group the number of indecomposable quasi-projective characters as defined in section 2 is finite. In the case of a block with cyclic defect group we characterize the indecomposable quasi-projective characters explicitly.

2. Quasi-projective and quasi-liftable characters

We call a quasi-projective character χ *decomposable* if χ is the sum of at least two non-zero quasi-projective characters. Otherwise χ is called *indecomposable.* Note that the set of irreducible ordinary characters allows a partition into p-blocks and an arbitrary character belongs to such a block, say B, if all its irreducible constituents belong to B.

Lemma 2.1. *An indecomposable quasi-projective character belongs to a block.*

Proof. This follows from [6, Ch. IV, Lemma 3.14]. □

Lemma 2.2. *If Λ is a quasi-projective character then either Λ belongs to a block of defect zero or Λ has at least two different irreducible constituents.*

Proof. Suppose that all irreducible constituents of Λ are equal. Hence $\Lambda = n\chi$ for some $n \in \mathbb{N}$ and $\chi \in \mathrm{Irr}(G)$. In particular, χ vanishes on the set of p-singular elements. Thus χ is also quasi-projective. In particular, $|G|_p \mid \chi(1)$ since $|G|_p \mid \Phi_\varphi(1)$ for all $\varphi \in \mathrm{IBr_p(G)}$, by

[10, Ch. VII, Corollary 7.16]. As χ is irreducible it belongs to a p-block of defect 0, by [14, Theorem 3.18]. □

The following example shows that the conclusion of Lemma 2.2 can not be improved and that the decomposition of a quasi-projective character into a sum of indecomposable quasi-projective characters is not unique.

Example 2.3. Let $G = A_5$ be the alternating group on 5 letters and let $p = 2$. Let φ and λ be the two irreducible 2-Brauer characters of G and let χ, ψ, ρ are the irreducible ordinary characters of degree $3, 3$ and 5 respectively. If we order the ordinary characters as $1, \chi, \psi, \rho$ and the Brauer characters as $1, \varphi, \lambda$ we may suppose that the decomposition matrix has shape

$$\begin{pmatrix} 1 & . & . \\ 1 & 1 & . \\ 1 & . & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

From this we get the decompositions

$$\begin{aligned} \Phi_1 &= (\Phi_1 - \Phi_\varphi) + \Phi_\varphi \\ &= (\Phi_1 - \Phi_\lambda) + \Phi_\lambda \end{aligned}$$

with quasi-projective summands

$$\begin{aligned} \Phi_1 - \Phi_\varphi &= 1 + \psi \\ \Phi_1 - \Phi_\lambda &= 1 + \chi \\ \Phi_\varphi &= \chi + \rho \\ \Phi_\lambda &= \psi + \rho. \end{aligned}$$

All these factors belong to the principal 2-block. Furthermore, they are indecomposable quasi-projective, for if not, then we would obtain an irreducible quasi-projective character, which would belong to a 2-block of defect zero (see proof of Lemma 2.2) This contradicts the fact that it lies in the principal 2-block.

The next lemma slightly generalizes [14, Lemma 10.16].

Lemma 2.4. *Let $\Lambda = \sum_{\varphi \in \mathrm{IBr_p(G)}} a_\varphi \Phi_\varphi$ be a quasi-projective character of G. If φ is quasi-liftable, then $a_\varphi \geq 0$.*

In particular, if all φ's with $a_\varphi \neq 0$ are quasi-liftable then Λ is projective.

Proof. By assumption, there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi^\circ = b\varphi$ for $b \in \mathbb{N}$. Clearly, the character $\Psi = b\Lambda = \sum_{\varphi \in \mathrm{IBr_p(G)}} ba_\varphi \Phi_\varphi$ is quasi-projective. Furthermore, $ba_\varphi = (\Psi, \varphi)^\circ$ by [14, Theorem 2.13]. It follows that

$$ba_\varphi = (\Psi, \varphi)^\circ = (b\Lambda, \varphi)^\circ = (\Lambda, \chi) \geq 0,$$

hence $a_\varphi \geq 0$. □

Lemma 2.5. *If there exists a $\varphi \in \mathrm{IBr_p(G)}$ which is not quasi-liftable then there exists a quasi-projective character of G (in the same p-block as φ) which is not projective.*

Proof. Suppose that φ is not quasi-liftable. Thus, for each $\chi \in \mathrm{Irr}(G)$ with $d_{\chi,\varphi} \neq 0$ there exists a $\psi \in \mathrm{IBr_p(G)}$ such that $\varphi \neq \psi$ and $d_{\chi,\psi} \neq 0$.

Let $b = \max\{d_{\chi,\varphi} \mid \chi \in \mathrm{Irr}(G)\}$. We consider $\Phi = -\Phi_\varphi + b \cdot \sum_{\psi \neq \varphi} \Phi_\psi$, where the sum is running over all $\psi \in \mathrm{IBr_p(G)}$ different from φ. The multiplicity of every $\chi \in \mathrm{Irr}(G)$ in Φ is given by

$$-d_{\chi,\varphi} + b \cdot \sum_{\varphi \neq \psi \in \mathrm{IBr_p(G)}} d_{\chi,\psi} \geq 0,$$

since there is a $\psi \in \mathrm{IBr_p(G)}$ with $\varphi \neq \psi$ and $d_{\chi,\psi} \geq 1$. This shows that Φ is a non-projective character. In case φ lies in the p-block B we take in the sum defining Φ only those ψ which lie in B. □

The last two lemmata prove Theorem 1.4. Together with Lemma 2.1 we see that the Theorem allows a block version as well.

3. Groups with cyclic Sylow p-subgroups

By [6, Ch. VII, Lemma 5.7], all decomposition numbers of a p-block B with a cyclic defect group are 1 or 0. This implies that every quasi-liftable p-Brauer character of B is liftable.

We have the following result of a general nature. For the definition and facts of Brauer trees the reader is referred to [1, Ch. V, section 17], [6, Ch. VII].

Lemma 3.1. *Let B be a p-block of G of positive defect with cyclic defect group and let Γ be the Brauer tree of B. Then the following are equivalent:*

a) *Every quasi-projective character of B is projective.*
b) *Every irreducible p-Brauer character of B is liftable.*
c) *Γ is a star with rays all of length 1.*

Proof. The equivalence of a) and b) follows from Theorem 1.4 and the above mentioned fact that every quasi-liftable p-Brauer character in B is liftable. Clearly, c) implies b). Finally, suppose that b) holds true, but Γ is not given as in c). Then Γ contains a line with at least 4 points and the irreducible p-Brauer character in the middle of the line is not liftable. □

Proposition 3.2. (Hiss, see [9]) *Let $G \not\cong PSL(2,q)$ be a non-abelian simple group. Then there is a prime p dividing $|G|$ such that a Sylow p-subgroup of G is cyclic and the p-Brauer tree of the principal p-block of G is not a star.*

In Hiss' result the prime p may be chosen such that it does not divide the order of the Schur multiplier of G. Thus Proposition 3.2 has an obvious extension to quasi-simple groups G with $G/\mathrm{Z(G)} \not\cong PSL(2,\mathrm{q})$.

Lemma 3.3. (Srinivasan, see [11, Section 16.10, Theorem]) *All p-modular decomposition numbers of $SL(2,p^r)$ are 0 or 1.*

Corollary 3.4. *Every quasi-liftable $\varphi \in \mathrm{IBr}_p(SL(2,p^r))$ is liftable.*

Note that in Theorem 1.5 the assertions b) and c) are equivalent, by Hiss' thesis [7]. It is easy to check that part b) implies a). Thus the following Corollary completes the proof of Theorem 1.5.

Corollary 3.5. *Let G be a quasi-simple group. Suppose that for every prime p all irreducible p-Brauer characters are quasi-liftable. Then $G \cong SL(2,5)$.*

Proof. If for some prime p a Brauer tree of G is not a star then there is a $\varphi \in \mathrm{IBr_p(G)}$ which is not quasi-liftable. Thus, according to the remark below Proposition 3.2 we get $G/Z(G) \cong PSL(2,q)$ for some q.
First we consider the case $G = SL(2,q)$:
We have to consider all primes p dividing $|SL(2,q)|$. For $p \mid q$ the quasi-liftable irreducible p-Brauer characters are liftable, by Lemma 3.4. Thus G has an irreducible complex character of degree 2 which means that G is a subgroup of $GL(2,\mathbb{C})$. By [5, Part A, Theorem 26.1], the group $SL(2,5)$ is the only non-solvable finite subgroup of $GL(2,\mathbb{C})$. Finally note that for all primes p dividing $|SL(2,5)|$ all irreducible p-Brauer characters are liftable.
Next we look at the simple groups $PSL(2,q)$:
According to Burkhard [3] and Corollary 3.4 all irreducible p-Brauer characters are liftable for all primes p since they are quasi-liftable by assumption. Thus we get $\varphi(1) \mid |G|$ for all $\varphi \in \mathrm{IBr_p(G)}$. In particular, G belongs to the class $\tilde{\mathcal{D}}$ in the notation of Hiss [8]. Thus, by [8, Theorem 4.1], the group G is isomorphic to $PSL(2,2^e) = SL(2,2^e)$ for some $e \geq 2$ or to $PSL(2,3^e)$ for $2 \leq e \leq 4$. Note that we dealt already with $SL(2,2^e)$. The latter three cases are ruled out by [13].
Finally, the remaining two groups $3.A_6$ and $6.A_6$ also do not satisfy the assumptions of the Corollary (see again [13]). □

4. Indecomposable quasi-projective characters

Let $\mathrm{Iqp}(G), \mathrm{Iqp}(B)$ denote the set of indecomposable quasi-projective characters of G resp. of a block B with respect to the prime p.

Theorem 4.1. $\mathrm{Iqp}(G)$ *is finite.*

Proof. Suppose there exists an infinite sequence $\Lambda_1, \Lambda_2, \ldots$ of different indecomposable quasi-projective characters. If $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$ we may write

$$\Lambda_i = \sum_{j=1}^{n} m_{ij}\chi_j \quad (m_{ij} \in \mathbb{N}_0).$$

We define $J \subseteq N = \{1, \ldots, n\}$ by $J = \{j \in N \mid \{m_{ij} \mid i = 1, 2, \ldots\} \text{ is finite}\}$. Note that J is a proper subset of N since the sequence $\Lambda_1, \Lambda_2, \ldots$ is infinite. Now we replace the sequence $\Lambda_1, \Lambda_2, \ldots$ by an infinite subsequence such that $|J|$ is maximal. We denote this subsequence again by $\Lambda_1, \Lambda_2, \ldots$. Observe that J does not change whenever we take an infinite subsequence of $\Lambda_1, \Lambda_2, \ldots$.

Next we write

$$\Lambda_i = \sum_{j \in J} m_{ij}\chi_j + \sum_{j \in N \setminus J \neq \emptyset} m_{ij}\chi_j.$$

If we put

$$\Lambda_i|_J = \sum_{j \in J} m_{ij}\chi_j$$

then we can choose an infinite subsequence of $\Lambda_1, \Lambda_2, \ldots$ which we denote again by $\Lambda_1, \Lambda_2, \ldots$ such that

$$\Lambda_i|_J = \Lambda_1|_J \quad \text{for all } i = 1, 2, \ldots.$$

Hence

$$\Lambda_i = \Lambda_1|_J + \sum_{j \in N \setminus J} m_{ij}\chi_j \quad \text{for } i = 1, 2, \ldots$$

Now we fix

$$\Lambda_1 = \Lambda_1|_J + \sum_{j \in N \setminus J} m_j\chi_j$$

where $m_j = m_{1j}$ and denote Λ_1 by Λ_1'.

Let $N \setminus J = \{j_1, \ldots, j_l\}$. In a first step we can find an infinite subsequence of $\Lambda_1, \Lambda_2, \ldots$ which we denote again by $\Lambda_1, \Lambda_2, \ldots$ such that

$$m_{ij_1} > m_{j_1} \quad \text{for all } i.$$

Taking again an infinite subsequence we get

$$m_{ij_2} > m_{j_2} \quad \text{for all } i.$$

Inductively we end up with an infinite sequence $\Lambda_1, \Lambda_2, \ldots$ such that

$$m_{ij_k} > m_{j_k} \quad \text{for all } i \text{ and } k = 1, \ldots, l.$$

By construction $\Lambda_i - \Lambda_1'$ is a quasi-projective character for all i which contradicts the fact that Λ_i is indecomposable. □

Question 4.2. Can one characterize $|\operatorname{Iqp}(G)|$ in terms of invariants of G or at least give a concrete upper bound?

Lemma 4.3. *For a p-block B we always have $|\operatorname{Iqp}(B)| \geq |\mathrm{IBr_p(B)}|$.*

Proof. Suppose that $\operatorname{Iqp}(B) = \{\Lambda_1, \ldots, \Lambda_r\}$ has size r. Since for $\varphi \in \mathrm{IBr(B)}$ each Φ_φ is a sum of indecomposable quasi-projective characters $\Lambda_1, \ldots, \Lambda_r$ generate the complex vector space of class functions generated by $\{\Phi_\varphi \mid \varphi \in \mathrm{IBr_p(B)}\}$. But this space has dimension $|\mathrm{IBr_p(B)}|$ according to [14, Theorem 2.13]. □

Lemma 4.4. *If all irreducible characters of a p-block B are quasi-liftable then $|\operatorname{Iqp}(B)| = |\mathrm{IBr_p(B)}|$.*

Proof. This is obvious since all quasi-projective characters of B are projective. Hence $\operatorname{Iqp}(B) = \{\Phi_\varphi \mid \varphi \in \mathrm{IBr_p(B)}\}$. □

The next example shows that the converse of the above lemma is not true in general.

Example 4.5. Let $G = \mathrm{A}_7$ and let B_0 denote the principal 3-block of G. According to (decomposition matrices in [13]) the decomposition matrix is given by

$$\begin{pmatrix} 1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ 1 & . & . & 1 \\ 1 & . & . & 1 \\ 2 & 1 & 1 & 1 \end{pmatrix}.$$

Clearly, the characters Φ_2, Φ_3, Φ_4 and $\Phi_1 - \Phi_4$ are indecomposable quasi-projective since all of them have at most three irreducible characters. We show that these are all indecomposable quasi-projective characters. Suppose that Λ is indecomposable quasi-projective, but not projective. Thus

$$\Lambda = a_1\Phi_1 + a_2\Phi_2 + a_3\Phi_3 - a_4\Phi_4$$

with $a_i \geq 0$ for all i and $a_4 > 0$. Since Λ is a character we get $a_1 \geq a_4$. Thus

$$\Lambda = (a_1\Phi_1 - a_4\Phi_4) + a_2\Phi_2 + a_3\Phi_3$$

and Λ indecomposable forces $a_2 = a_3 = 0$. Finally $a_1 = a_4 = 1$ since otherwise Λ is decomposable.

Question 4.6. Can one characterize blocks B with $|\operatorname{Iqp}(B)| = |\mathrm{IBr_p(B)}|$?

Theorem 4.7. *Let B be a p-block of positive defect with a cyclic defect group. By χ_0 we denote the sum of exceptional irreducible characters of B (if such characters exist). Furthermore let $\mathrm{Irr}^0(B)$ be the set consisting of χ_0 and all the non-exceptional irreducible characters of B. Then*

$$\Lambda = \sum_{\varphi \in \mathrm{IBr_p(B)}} a_\varphi \Phi_\varphi$$

is an indecomposable quasi-projective character of B if and only if $\Lambda = \chi + \psi$ for $\chi, \psi \in \mathrm{Irr}^0(B)$ where the distance between χ and ψ in the Brauer tree is odd.

Proof. In the following we use several times the fact that $\Phi_\varphi = \chi + \psi$ for $\chi, \psi \in \mathrm{Irr}^0(B)$ where χ and ψ have distance 1 in the Brauer tree (see [6, Ch. VII, Theorem 2.19]).

We write $\Lambda = \sum_{\chi \in \mathrm{Irr}^0(B)} b_\chi \chi$ and assume that Λ is indecomposable quasi-projective. Let $\mathrm{d}(\cdot\,,\cdot)$ denote the distance in the Brauer tree. Furthermore, let $\chi, \psi \in \mathrm{Irr}^0(G)$ with $\mathrm{d}(\chi, \psi) = d$ odd and $b_\chi \neq 0 \neq b_\psi$. We consider the path from χ to ψ which is by definition of length d, and unique. Associated to this path, say

$$\chi = \eta_1, \eta_2, \ldots, \eta_{d+1} = \psi,$$

there are p-Brauer characters

$$\varphi_1, \ldots, \varphi_d$$

where φ_i is contained in the constant reduction of η_i and η_{i+1} We obtain from this the quasi-projective character

$$\Psi = \Phi_{\varphi_1} - \Phi_{\varphi_2} + \ldots - \Phi_{\varphi_{d-1}} + \Phi_{\varphi_d} = \chi + \psi.$$

Hence $\Lambda = \Psi$ since Λ is indecomposable and we are done.

Thus we have the following. Either $\Lambda = \chi + \psi$ and the distance between χ and ψ is odd or $\Lambda = \sum_{\chi \in \mathrm{Irr}^0(B)} b_\chi \chi$ where $b_\chi \neq 0 \neq b_\psi$ if and only if the path between χ and ψ is of even length.

Next we show that the latter case does not occur. If $g \in G$ is p-singular then

$$0 = \Phi_{\varphi_i}(g) = \eta_i(g) + \eta_{i+1}(g), \qquad (*)$$

hence $\eta_i(g) = -\eta_{i+1}(g) = c_g$.

We have to deal with the case that the distance between two characters of $\mathrm{Irr}^\circ(B)$ is even whenever they occur in Λ. If $\chi \in \mathrm{Irr}(B)$ then there is a p-singular $g \in G$ with $\chi(g) = c \neq 0$ since χ is not of p-defect zero. The same holds true for χ_0. This can be seen as follows. There exists a $\varphi \in \mathrm{IBr_p(G)}$ with $a_\varphi \neq 0$ and $\Phi_\varphi = \psi + \chi_0$ where $\psi \in \mathrm{Irr}(B)$. If $\chi_0(g) = 0$ for all p-singular elements $g \in G$ then the same holds true for the irreducible character ψ. This implies that ψ is of p-defect zero, a contradiction.

Now we fix some $\chi \in \mathrm{Irr}^0(B)$ which occurs in Λ. Let ψ be any other character in Λ which is irreducible or χ_0. Since the distance between χ and ψ is even we get $\chi(g) = \psi(g) = c$ according to $(*)$. Thus $\Lambda(g) = nc \neq 0$ where $n = \sum_{\chi \in \mathrm{Irr}^0(B)} b_\chi$, a contradiction.

Thus we have proved that $\Lambda = \chi + \psi$ for $\chi, \psi \in \mathrm{Irr}^0(B)$ with $\mathrm{d}(\chi, \psi)$ odd if Λ is indecomposable quasi-projective. Conversely any character $\Lambda = \chi + \psi$ with $\chi, \psi \in \mathrm{Irr}^0(B)$ and $\mathrm{d}(\chi, \psi)$ odd is quasi-projective according to $(*)$. Moreover it is indecomopsable. This is obvious if χ and ψ are irreducible characters. Thus suppose that $\chi = \chi_0$ and $\Lambda = \Lambda_1 + \Lambda_2$ with $\Lambda_i \neq 0$ and quasi-projective. In this case we may suppose that Λ_1 is the sum of some exceptional irreducible characters and Λ_2 contains ψ. Since a projective character contains all exceptional characters or none we see that $\Lambda_1 = \chi_0$. This implies that $\Lambda_2 = \psi$ is of p-defect zero, a contradiction.

□

Theorem 4.7 shows that the ordinary characters of all projective indecomposable modules (so-called PIM characters) are indecomposable quasi-projective. Thus we may ask:

Question 4.8. Can one characterize all groups for which every PIM character is indecomposable quasi-projective? Note that such a classification must include all groups in which every quasi-projective character is projective. !

Apart from blocks with cyclic defect groups there are other non-trivial examples for which a complete classification of indecomposable quasi-projective characters is achievable. For the following the reader is refered to [12, section 28].

Let $G = \mathrm{GL}(2, q)$ where the prime p divides q. For $\zeta \in \mathrm{Irr}(Z(G))$ we denote by $R_1(\zeta)$, $R_{q-1}(\zeta)$ and $R_{q+1}(\zeta)$ the sets $\chi \in \mathrm{Irr}(G)$ of degree $1, q-1$ and $q+1$, respectively, such that $\chi|_{Z(G)} = \chi(1)\zeta$. Let $z \in Z(G)$ and $1 \neq u \in U$, where U is a Sylow p-subgroup of G. Note that $C_G(u) = Z(G)U$. Using the character table of G, one observes that $\chi(u) = 1$ if $\chi \in R_1(\zeta) \cup R_{q+1}(\zeta)$, and $\chi(u) = -1$ if $\chi \in R_{q-1}(\zeta)$. Furthermore, the p-singular elements are conjugate to elements of shape zu where $z \in Z(G)$ and $1 \neq u \in U$. Therefore, $\chi(zu) = \zeta(z)\chi(u)$. It follows that $\chi(zu) = \zeta(z)$ if $\chi \in R_1(\zeta) \cup R_{q+1}(\zeta)$, and $\chi(zu) = -\zeta(z)$ if $\chi \in R_{q-1}(\zeta)$. Thus $\tau + \sigma$ is quasi-projective whenever one of the following holds:

(1) $\tau \in R_1(\zeta)$ and $\sigma \in R_{q-1}(\zeta)$ or conversely,

(2) $\tau \in R_{q-1}(\zeta)$ and $\sigma \in R_{q+1}(\zeta)$ or conversely.

Finally note that the characters $\tau + \sigma$ with property (1) or (2) are even indecomposable quasi-projective. All irreducible characters of G which do not belong to some $R_i(\zeta)$ are of defect zero and degree q.

Proposition 4.9. *Every indecomposable quasi-projective character of* $G = \mathrm{GL}(2, q)$ *with* $p \mid q$ *is either irreducible of degree q, or as in* (1)*, or as in* (2)*.*

Proof. Let $\Lambda = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$ be an indecomposable quasi-projective character of G. We know (Lemma 2.1) that all χ with $a_\chi \neq 0$ belong to the same block. This implies that all these characters belong to $R_1(\zeta) \cup R_{q-1}(\zeta) \cup R_{q+1}(\zeta)$ for the same (fixed) ζ. Clearly, we may assume that no constituent χ is of degree q. The minimality of Λ implies that no pair of characters satisfying (1) or (2) can occur in this decomposition, otherwise we are done. Furthermore, $\Lambda(u) \neq 0$ for every p-element $u \in G$ if all irreducible characters χ with $a_\chi \neq 0$ belong to the same $R_i(\zeta)$. Since no χ in Λ with $a_\chi \neq 0$ can belong to $R_{q-1}(\zeta)$ (otherwise Λ has a summand of type (1) or (2) and we are done), all χ ocurring in Λ are in $R_1(\zeta) \cup R_{q+1}(\zeta)$. This implies that $\Lambda(u) > 0$ for all p-elements $u \in G$, a contradiction.

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Wolfgang Willems
Otto-von-Guericke University, Magdeburg, Germany
and Universidad del Norte, Barranquilla, Colombia

Alexander Zalesski
National Academy of Sciences of Belarus, Minsk, Belarus
and University of East Anglia, Norwich, UK