Detecting and Reconstructing Centrally Symmetric Sets from the Autocorrelation: Two Discrete Cases

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Abstract

Let $A$ be a finite, centrally symmetric set in $\mathbb{R}^d$, $d \geq 1$, and let $B$ be a set homometric to $A$, that is, for all $x \in \mathbb{R}^d$ the sets $A \cap (A + x)$ and $B \cap (B + x)$ have equal cardinalities. We show that $B$ is a translate of $A$. As a consequence, an analogous statement is obtained for bodies which are unions of lattice cubes. We notice that homometric sets are relevant in several research areas including image analysis, geometric probability, and X-ray crystallography.

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1 Introduction

The problem of reconstructing a given object (e.g., set, function, or distribution) from the autocorrelation is relevant in several research areas, such as geometric tomography [Gar06, p. 378], [HK07], image analysis [Ser84, Ch. IX], geometric probability [Mat75, §4.3], phase retrieval [Hur89], and X-ray crystallography [Jan97]. Usually it can be shown that a centrally symmetric object can be reconstructed from the autocorrelation within a given class of centrally symmetric objects. A more general problem is concerned with detection of central symmetry within a class of objects which are not all centrally symmetric and subsequent reconstruction of a centrally symmetric object. The purpose of this note is to study the above problem for finite sets and for unions of lattice cubes.

Let $A$ be a finite subset of $\mathbb{R}^d$, $d \geq 1$. Then the function $g_A(x) := |A \cap (A + x)|$, where $x \in \mathbb{R}^d$ and $| \cdot |$ stands for cardinality, is called the autocorrelation (or discrete covariogram) of $A$. Clearly, $g_A$ is invariant with respect to reflections and translations of $A$. In general it is not possible to determine $A$ from $g_A$, up to translations and reflections, even under extra (regularity) assumptions on $A$ such as (discrete versions of) convexity of $A$. Fig. 1 depicts two convex lattices sets with equal autocorrelation, where the example is borrowed from [GGZ05, p. 402].

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In contrast to the above remark, we can determine arbitrary finite, centrally symmetric sets within the class of all finite sets.

**Theorem 1.** Let $A$ be a finite, centrally symmetric set in $\mathbb{R}^d$, $d \geq 1$, and let $B$ be a finite set with $g_A = g_B$. Then $B$ is a translate of $A$.

For a compact set $K$ in $\mathbb{R}^d$ with non-empty interior the autocorrelation (or continuous covariogram) of $K$ is the function $g_K(x) = \lambda_d(K \cap (K+x))$, where $\lambda_d$ is the $d$-dimensional Lebesgue measure. Let us call $K$ a lattice body if $K = A + [0,1]^d$ for some finite set $A$ in $\mathbb{Z}^d$. (Notice that lattice bodies are generalizations of polyominoes.) Positive results on determination of convex sets from autocorrelation have recently been obtained in [AB] and [Bia] (see also the references therein). However, also in the case of compact sets with non-empty interior in general it is not possible to determine a set from its autocorrelation, up to translations and reflections. In fact, the sets $K = A + [0,1]^2$ and $H = B + [0,1]^2$ with $A$ and $B$ as in Fig. 1 do not coincide, up to translations and reflections, and satisfy $g_K = g_H$ (see [GGZ05] for more details). As a consequence of Theorem 1 we show that a centrally symmetric lattice body $K$ can be determined within the class of all lattice bodies by $g_K$.

**Corollary 2.** Let $K$ and $H$ be two lattice bodies with $g_K = g_H$ and let $K$ be centrally symmetric. Then $H$ is a translate of $K$.

Following referee’s suggestion, we notice that the results of this paper were motivated by Proposition 4.4 and Corollary 4.5 in [GGZ05], which show that, within the class of centrally symmetric sets, $X$ can be recovered from $g_X$, up to translations, for the case when $X$ is finite and for the case when $X$ is a compact set in $\mathbb{R}^d$ coinciding with the closure of its interior. In fact, Proposition 3 below gives a common generalization of both of these cases in terms of distributions (see also [KST95, Proposition 3.3] for an analogous statement for $d = 1$ and [Höhr03] for basic information on distributions). A distribution $f$ on $\mathbb{R}^d$ is said to be centrally symmetric if $f \circ R = f$ for some point reflection $R$ in $\mathbb{R}^d$ and non-negative if $f$ takes non-negative values on non-negative test functions. We remark that, essentially by the Riesz representation theorem (see [Tay06, Chapter 13]), non-negative distributions are in one-to-one correspondence with (non-negative) Radon measures. The Fourier transform of $f$ is denoted by $\hat{f}$.
Proposition 3. Let $f$ and $h$ be non-negative, centrally symmetric distributions on $\mathbb{R}^d$ having compact supports and satisfying $|\hat{f}| = |\hat{h}|$. Then $f \circ T = h$ for an appropriate translation $T$.

Motivated by the results presented above, we ask the following question. Let $K$ and $H$ be compact sets equal to the closure of their interior and let $K$ be centrally symmetric. Does the condition $g_K = g_H$ imply that $K$ and $H$ coincide, up to translations? In this respect we notice that the proof idea of Proposition 3 does not seem to be directly extendable to the case when $h$ is not assumed to be centrally symmetric. On the other hand, by considering various homothetic copies of lattice bodies, Corollary 2 provides a positive answer to the above question for a class of bodies which is “arbitrarily dense” in the class of all compact sets.

2 The proofs

By $o$ we denote the zero-vector in $\mathbb{R}^d$. The support is abbreviated as supp. By $[a, b]$ we denote the line segment joining two points $a$ and $b$ in $\mathbb{R}^d$. We remark that for a finite set $A$ in $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, $g_A(x)$ is precisely the number of pairs $(x_0, x_1)$ with $x_0, x_1 \in A$ and $x_1 - x_0 = x$.

Proof of Theorem 1. We start with the case $d = 1$. Let us exclude the trivial situation $|A| \leq 1$. Let $g := g_A = g_B$ and let $y > 0$ be the maximal value in supp $g$. After replacing $A$ and $B$ by appropriate translates we may assume that both $A$ and $B$ are contained in $[0, y]$. We have $[y/2, y] \cap A = \{y_1, \ldots, y_m\}$ with $m \in \mathbb{N}$ and appropriate $y_1 < \ldots < y_m = y$. By construction, 0 and $y = y_m$ belong to both $A$ and $B$. We show by (the reverse) induction that the sets $A_k := ([0, y - y_k] \cup [y_k, y]) \cap A$ and $B_k := ([0, y - y_k] \cup [y_k, y]) \cap B$ coincide for every $k = 1, \ldots, m$. For $k = m$ that was noticed above. Now assume that $A_{k+1}$ and $B_{k+1}$ coincide, where $k = 1, \ldots, m - 1$. Then every $x$ with $y_k < x < y_{k+1}$ does not belong to $B$. In fact, assume the contrary and fix $x$ with $y_k < x < y_{k+1}$ and $x \in B$. Then all pairs $(x_0, x_1)$ with $x_0, x_1 \in A$ and $x_1 - x_0 = x$ satisfy $x_0, x_1 \in A_{k+1} = B_{k+1}$. Furthermore, $B$ possesses at least one further pair $(x_0, x_1)$ with $x_0, x_1 \in B$ and $x_1 - x_0 = x$, since we may set $x_0 = 0$ and $x_1 = x$. Hence $g_B(x) > g_A(x)$, a contradiction. Analogously we can show that every $x$ with $y < y_{k+1} < x < y_k$ does not belong to $B$. Next we show that \{y_k, y - y_k\} is a subset of $B$. We have

$$g_A(y_k) = |\{(x_0, x_1) : x_0, x_1 \in A, x_1 - x_0 = y_k\}|$$
$$= |\{(x_0, x_1) : x_0, x_1 \in A_{k+1}, x_1 - x_0 = y_k\}| + |\{(0, y_k), (y - y_k, y)\}|$$
$$= |\{(x_0, x_1) : x_0, x_1 \in B_{k+1}, x_1 - x_0 = y_k\}| + |\{(0, y_k), (y - y_k, y)\}|$$
$$\geq |\{(x_0, x_1) : x_0, x_1 \in B_{k+1}, x_1 - x_0 = y_k\}| + |\{(0, y_k), (y - y_k, y)\} \cap (B \times B)|$$
$$= g_B(y_k) = g_A(y_k).$$

Thus, $A_{k+1} = B_{k+1}$ implies $A_k = B_k$ and we obtain the equality $A_1 = B_1$. Using analogous arguments we show that no $x$ with $y - y_1 < x < y_1$ belongs to $B$. Summarizing we get $A = B$.

For an arbitrary $d \in \mathbb{N}$, we argue by induction on $d$. For $d = 1$, the statement was shown above. Assume that the statement of the theorem is valid for dimension $d - 1$ with $d \geq 2$. Let $A$ and $B$ be the sets satisfying the assumptions of the theorem. Without loss
of generality let the center of mass of both $A$ and $B$ be the origin. For a unit vector $u$ in $\mathbb{R}^d$ we denote by $H_u$ the plane through the origin orthogonal to $u$, and by $T_u$ the operator of orthogonal projection onto $H_u$. Let $U$ be the set of all unit vectors $u$ in $\mathbb{R}^d$ such that $u$ is not parallel to any vector $x_1 - x_0$ with $x_0 \neq x_1$ and $x_0, x_1 \in \text{supp} \ g$. For every $u \in U$ the mapping $T_u$ is injective on the sets $A$, $B$, and $\text{supp} \ g$. Clearly, $U$ is a relatively open subset of the unit sphere. For all $u \in U$ we obviously have $T_u(\text{supp} \ g) = \text{supp} \ g_{T_uA} = \text{supp} \ g_{T_uB}$. Furthermore, for $u \in U$ and $x \in \text{supp} \ g$

$$g_{T_uB}(T_u x) = \left\{ (T_u x_0, T_u x_1) : x_0, x_1 \in B, T_u x_1 - T_u x_0 = T_u x \right\}$$

$$= \left\{ (T_u x_0, T_u x_1) : x_0, x_1 \in B, x_1 - x_0 = x \right\}$$

(by injectivity of $T_u$ on $\text{supp} \ g$)

$$= \left\{ (x_0, x_1) : x_0, x_1 \in B, x_1 - x_0 = x \right\}$$

(by injectivity of $T_u$ on $B$)

$$= g_B(x) = g_A(x) = g_{T_uA}(T_u x).$$

Thus, $g_{T_uA}(x) = g_{T_uB}(x)$ for all $u \in U$ and $x \in H_u$. Applying the induction assumption to $T_uA$ and $T_uB$ and taking into account that for all $u \in U$ the origin is the center of mass of $T_uA$ and $T_uB$, we get $T_uA = T_uB$ for all $u \in U$. Then, in view of the result from [Hep56], stating that a finitely supported measure on $k$ points can be determined by $k + 1$ X-ray pictures in mutually non-parallel directions, we obtain $A = B$. \hfill \square

In the proofs of Corollary 2 and Proposition 3 we use elements of the theory of distributions. As usual $\delta_a$ denotes the delta-function concentrated at $a$. For a finite set $A$ in $\mathbb{R}^d$ we define $\delta_A := \sum_{a \in A} \delta_a$. The convolution is denoted by $\ast$. The characteristic function of a set $K$ is denoted by $1_K$. If $K$ is a compact set with non-empty interior we easily get the equality $g_K = 1_K \ast 1_{-K}$. Analogously, for a finite set $A$ there is a one-to-one correspondence between $g_A$ and $\delta_A \ast \delta_{-A}$ in view of $\sum_{a \in \text{supp} \ g_A} g_A(a) \delta_a = \delta_A \ast \delta_{-A}$.

**Proof of Corollary 2.** Let us represent $K$ and $H$ in the form $K = A + C$ and $H = B + C$, where $A, B$ are finite, $C = [0, 1]^d$, and $A$ is centrally symmetric. It is easy to check that $1_K = \delta_A \ast 1_C$. Hence $g_K = \delta_A \ast \delta_{-A} \ast 1_C \ast 1_{-C} = \delta_A \ast \delta_{-A} \ast g_C$. Analogously, $g_H = \delta_B \ast \delta_{-B} \ast g_C$.

Applying the Fourier transform to $g_K = g_H$ we obtain $\hat{g}_K = |\hat{\delta}_A|^2 \hat{g}_C = |\hat{\delta}_B|^2 \hat{g}_C = \hat{g}_H$. It is well known that the Fourier transform of a distribution with compact support is an analytic function. Choose an open subset $U$ of $\mathbb{R}^d$ on which the analytic function $\hat{g}_C$ does not vanish. It follows that $|\hat{\delta}_A|^2 = |\hat{\delta}_B|^2$ in $U$. By analyticity of $|\hat{\delta}_A|^2$, we get $|\hat{\delta}_A|^2 = |\hat{\delta}_B|^2$ in the whole $\mathbb{R}^d$. Applying the reverse Fourier transform to the latter relation, we arrive at $\delta_A \ast \delta_{-A} = \delta_B \ast \delta_{-B}$. Hence $g_A = g_B$, and in view of Theorem 1, $B$ is a translate of $A$. It follows that $H$ is a translate of $K$. \hfill \square

**Proof of Proposition 3.** Essentially, we may directly extend the short proof of Proposition 4.4 from [GGZ05]. Without loss of generality let $f = f \circ R$ and $h = h \circ R$, where $R$ is the reflection in the origin. Taking into account the above symmetry relations, the equality $|\hat{f}| = |\hat{h}|$ yields $|\hat{f}|^2 = (\hat{h})^2$. Hence $\hat{f}(x) = \pm \hat{h}(x)$ for all $x \in \mathbb{R}^d$ with the sign, a priori, depending on $x$. However, since both $\hat{f}$ and $\hat{h}$ are analytic, they are uniquely determined by their values on a set with a limit point, so that we obtain $\hat{f} = \pm \hat{h}$. Applying the inverse Fourier transform and using non-negativity of $f$ and $h$, we get the assertion. \hfill \square
References


