

# ON HILBERT DIVISORS OF BRAUER CHARACTERS

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**Abstract** The Hilbert divisor  $p^{a(\varphi)}$  of an irreducible  $p$ -Brauer character  $\varphi$  of a finite group  $G$  carries deep information about  $\varphi$ , respectively the module affording  $\varphi$ . In [8] we conjectured that  $\varphi$  belongs to a  $p$ -block of defect 0 if and only if its Hilbert divisor is 1. In this note we continue our investigations.

**Keywords:** block, defect, Cartan matrix, Brauer character, Hilbert divisor, Green correspondent

**MSC2010:** 20C20

## 1. INTRODUCTION

Throughout the paper  $p$  is always a prime and  $G$  a finite group. We put  $|G|_p = p^a$  and

$$G_{p'} = \{g \mid g \in G, g \text{ is a } p'\text{-element}\}.$$

By  $\text{IBr}_p(G)$  and  $\text{IBr}_p(B)$  we denote the set of irreducible  $p$ -Brauer characters of  $G$ , resp. of a  $p$ -block  $B$  of  $G$  with respect to a  $p$ -modular splitting system  $(K, R, k)$ . Here  $R$  is a complete discrete valuation ring with unique maximal ideal  $\pi R$ ,  $K$  is the quotient field of  $R$  of characteristic 0 and  $k = R/\pi R$  the residue field of characteristic  $p > 0$ .

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Furthermore, let  $R^* = R \setminus \pi R$  be the set of units in  $R$ . Similarly, we define  $\text{Irr}(G)$ , resp.  $\text{Irr}(B)$  as the set of irreducible complex characters of  $G$ , resp.  $B$ . We write  $\Phi_\varphi$  for the ordinary character associated to the projective cover of the module affording  $\varphi \in \text{IBr}_p(G)$ . To be brief we call the ordinary character of a projective module a projective character. If  $\chi$  is a generalized ordinary character of  $G$ , then  $\chi^\circ$  denotes the restriction of  $\chi$  on  $G_{p'}$ . Let  $vx(\varphi)$  denote the vertex of the module which affords  $\varphi$ . Finally, if  $\varphi \in \text{IBr}_p(B)$  where the block  $B$  has defect  $d$ , then the height  $\text{ht}(\varphi)$  of  $\varphi$  is defined by  $\varphi(1)_p = p^{a-d+\text{ht}(\varphi)}$ . Note that in contrast to ordinary irreducible characters  $\text{ht}(\varphi) > d$  may happen.

In [8] we defined Hilbert divisors for irreducible Brauer characters, i.e., if  $\varphi \in \text{IBr}_p(G)$ , then the Hilbert divisor of  $\varphi$  is the minimal  $p$ -power, say  $p^{a(\varphi)}$ , such that  $p^{a(\varphi)}\varphi$  is a quasi-projective Brauer character which means that

$$p^{a(\varphi)}\varphi = \sum_{\psi \in \text{IBr}_p(G)} a_\psi \Phi_\psi^\circ \text{ with } a_\psi \in \mathbb{Z}. \quad (1)$$

If  $\varphi$  lies in the block  $B$  of defect  $d$  and  $|vx(\varphi)| = p^v$ , then the exponents  $a(\varphi)$  of the Hilbert divisors satisfy  $a(\varphi) \leq v \leq d$  and  $a(\varphi) = d$  if  $\text{ht}(\varphi) = 0$  ([8], Theorem 2.1). According to Dickson's Theorem ([11], Corollary 2.14) we have  $p^a \mid \Phi_\psi(1)$  for all  $\psi \in \text{IBr}_p(G)$ . Thus from (1) we immediately get

- (i)  $d \leq a(\varphi) + \text{ht}(\varphi)$
- (ii)  $p^{a-a(\varphi)} \mid \varphi(1)$ .

Note that (ii) improves the well-known fact  $p^{a-v} \mid \varphi(1)$ . For instance, in the principal 2-block of the first simple Janko group  $J_1$  there is an irreducible Brauer character  $\varphi$  of degree 56 with Hilbert divisor  $2 = 2^{a(\varphi)}$ , but  $|vx(\varphi)| = 2^3$ .

One of the main problems left open in [8] is the following.

**Conjecture A.**  $\varphi \in \text{IBr}_p(G)$  belongs to a block of defect 0 if and only if its Hilbert divisor is 1, i.e.,  $a(\varphi) = 0$ .

The conjecture holds true in the following cases:

- a) for  $p$ -solvable groups ([8], Proposition 2.7),
- b) for  $p$ -blocks with cyclic defect groups ([8], Proposition 2.9),
- c) for blocks of tame representation type according to Erdmann's work [2].

## 2. BRAUER CHARACTERS OF HEIGHT ZERO

Throughout this section let  $B$  be a  $p$ -block of  $G$  of defect  $d$  and let  $\varphi \in \text{IBr}_p(B)$ .

**Lemma 2.1.** *If  $\text{ht}(\varphi) = 0$ , then  $a(\varphi) = d$ .*

*Proof.* This is ([8], Theorem 2.1 c).  $\square$

**Lemma 2.2.** *If  $G$  is  $p$ -solvable, then  $a(\varphi) + \text{ht}(\varphi) = d$ .*

*Proof.* The following equation

$$\begin{aligned} \Phi_\varphi(1) &= p^a \varphi(1)_{p'} && \text{(by ([3], Chap. X, Theorem 3.2))} \\ &= p^{a(\varphi)} \varphi(1) && \text{(by ([8], Proposition 2.8))} \\ &= p^{a(\varphi)+a-d+\text{ht}(\varphi)} \varphi(1)_{p'} \end{aligned}$$

implies  $a(\varphi) + \text{ht}(\varphi) = d$ .  $\square$

We would like to mention here that the conclusion in Lemma 2.2 very often, but not always holds true for groups which are not  $p$ -solvable.

**Corollary 2.3.** *For a  $p$ -solvable group we have  $a(\varphi) = d$  if and only if  $\text{ht}(\varphi) = 0$ .*

**Example 2.4.** Let  $G$  be the alternating group  $A_9$  of degree 9 and let  $B = B_0$  be the principal 3-block of  $G$ . According to [9] the block  $B$  contains five irreducible Brauer characters of degree 1, 7, 21, 35, 41. Note that only the third character has non-zero height. For  $a(\varphi)$  one easily computes 4, 4, 3, 4, 4.

**Question 2.5.** *Does Corollary 2.3 hold true without the assumption of  $p$ -solvability?*

**Lemma 2.6.** *If  $\varphi \in \text{IBr}_p(B)$ , then the following are equivalent.*

- a)  $p^{a(\varphi)} \varphi$  is of height 0.
- b)  $B$  is a block of defect 0.

*Proof.* If  $p^{a(\varphi)} \varphi$  is of height 0, then  $a - d = a(\varphi) + a - d + \text{ht}(\varphi)$ , hence  $a(\varphi) = 0 = \text{ht}(\varphi)$  and b) follows directly from ([8], Proposition 2.10). The other direction is clear.  $\square$

## 3. CHARACTERIZATION OF HILBERT DIVISORS

**Theorem 3.1.** *If  $\varphi \in \text{IBr}_p(G)$ , then*

$$\frac{p^{a(\varphi)} \varphi(x)}{|C_G(x)|_p} \in R$$

*for all  $x \in G_{p'}$  and there exists an  $x$  such that  $\frac{p^{a(\varphi)} \varphi(x)}{|C_G(x)|_p} \in R^*$ .*

*Proof.* Note that

$$p^{a(\varphi)}\varphi(x) = \sum_{\psi \in \text{IBr}_p(B)} a_\psi \Phi_\psi^\circ(x) \quad (a_\psi \in \mathbb{Z}, x \in G_{p'}),$$

by the definition of Hilbert divisors. Thus

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} = \sum_{\psi \in \text{IBr}_p(B)} a_\psi \frac{\Phi_\psi^\circ(x)}{|C_G(x)|_p}.$$

Now ([11], Lemma 2.21) says that

$$\frac{\Phi_\psi(x)}{|C_G(x)|_p} \in R$$

for  $x \in G_{p'}$  and the first part of the assertion has been proved. To get the second part we have to prove that

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \notin \pi R$$

for some  $x \in G_{p'}$ . Suppose that  $\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|_p} \in \pi R$  for all  $x \in G_{p'}$ . Thus, if  $\langle \cdot, \cdot \rangle^\circ$  denotes the usual scalar product of the  $K$ -class functions on  $G_{p'}$ , then

$$\langle p^{a(\varphi)}\varphi, \psi \rangle^\circ = \frac{1}{|G|} \sum_{x \in G_{p'}} p^{a(\varphi)}\varphi(x)\psi(x^{-1}) = \sum_{i=1}^h \frac{p^{a(\varphi)}\varphi(x_i)}{|C_G(x_i)|} \psi(x_i^{-1}) \in \pi R$$

for all  $\psi \in \text{IBr}_p(G)$ , a contradiction to the following lemma.  $\square$

**Lemma 3.2.** *Let  $\varphi \in \text{IBr}_p(B)$ . Then there exists a  $\psi \in \text{IBr}_p(B)$  such that*

$$\langle p^{a(\varphi)}\varphi, \psi \rangle^\circ \in R^*.$$

*Proof.* Let  $C^{-1} = (c^{\varphi\psi})$  denote the inverse of the Cartan matrix of  $B$ . Then

$$\langle p^{a(\varphi)}\varphi, \psi \rangle^\circ = p^{a(\varphi)}c^{\varphi\psi}.$$

By the construction of the Hilbert divisors (see part a) of the proof of ([8], Theorem 2.1)) we see that  $p^{a(\varphi)}c^{\varphi\psi} \in \mathbb{Z}$  for all  $\psi \in \text{IBr}_p(B)$  and there exists a  $\psi \in \text{IBr}_p(B)$  such that  $p \nmid p^{a(\varphi)}c^{\varphi\psi}$ .  $\square$

Let  $\hat{\cdot} : R \rightarrow R/\pi R = k$  denote the natural epimorphism. Note that for  $\varphi \in \text{IBr}_p(G)$  and  $x \in G_{p'}$  the value  $\widehat{\varphi(x)}$  is the trace of  $x$  with respect to the  $k$ -representation affording  $\varphi$ .

**Corollary 3.3.** *If  $\varphi \in \text{IBr}_p(G)$  and  $\widehat{\varphi(x)} \neq 0$  for  $x \in G_{p'}$ , then  $|C_G(x)|_p \mid p^{a(\varphi)}$ .*

*Proof.*  $\widehat{\varphi(x)} \neq 0$  means that  $\varphi(x) \in R^*$ . Thus Theorem 3.1 implies the assertion.  $\square$

**Example 3.4.** In general there is no  $x \in G_{p'}$  with  $|C_G(x)|_p = p^{a(\varphi)}$ . As an example we may take the alternating group  $G = A_5$  and  $p = 2$ . In this case there exists  $\varphi \in \text{IBr}_2(G)$  with  $\varphi(1) = 2$  and  $a(\varphi) = 1$ . On the other hand  $|C_G(x)|_2 = 1$  for all  $1 \neq x \in G_{2'}$ .

**Corollary 3.5.** *If  $\varphi \in \text{IBr}_p(G)$  with  $a(\varphi) = 0$ , then there exists an  $x \in G_{p'}$  of  $p$ -defect 0 such that  $\varphi(x) \in R^*$ .*

Theorem 3.1 immediately leads to the following characterization of Hilbert divisors.

**Theorem 3.6.** *If  $\varphi \in \text{IBr}_p(G)$ , then  $a(\varphi)$  is the smallest non-negative integer  $n$  such that  $\frac{p^n \varphi(x)}{|C_G(x)|_p} \in R$  for all  $x \in G_{p'}$  and there exists an  $x \in G_{p'}$  such that  $\frac{p^n \varphi(x)}{|C_G(x)|_p} \in R^*$ .*

With the Theorem above we can reformulate Conjecture A to a statement which obviously extends a classical result, namely that  $\chi \in \text{Irr}(G)$  belongs to a block of defect 0 if  $\frac{\chi(1)}{|C_G(1)|_p} \in R^*$ .

**Conjecture A\*.** If  $\varphi \in \text{IBr}_p(G)$ , then the following are equivalent.

- a)  $\varphi$  lies in a  $p$ -block of defect 0.
- b)  $\frac{\varphi(x)}{|C_G(x)|_p} \in R$  for all  $x \in G_{p'}$  and for at least one  $x$  we have  $\frac{\varphi(x)}{|C_G(x)|_p} \in R^*$ .

**Question 3.7.** *Do we always have*

$$\varphi(1)_p \leq p^{a+a(\varphi)}$$

*for  $\varphi \in \text{IBr}_p(G)$  where  $p^a = |G|_p$ ?*

**Remark 3.8.** If the bound in Question 3.7 holds true, then it is sharp. The McLaughlin group  $G = \text{McL}$  has an irreducible Brauer character  $\varphi$  in the principal 2-block with  $\varphi(1)_2 = 2^9$  (see [9]). Note that  $|G|_2 = 2^7$ . One easily computes  $a(\varphi) = 2$  which shows that the bound is sharp.

#### 4. DEFECT OF IRREDUCIBLE BRAUER CHARACTERS

Let  $B$  be a  $p$ -block of  $G$  of defect  $d$ . Then the defect  $d(\chi)$  of  $\chi \in \text{Irr}(B)$  is defined by  $d(\chi) = d - \text{ht}(\chi)$ . If we replace  $\chi$  by a Brauer character  $\varphi$ , this number may be negative. Thus, for Brauer characters we define the defect as follows.

**Definition 4.1.** If  $\varphi \in \text{IBr}_p(B)$ , then we call

$$d(\varphi) = a(\varphi) + \text{ht}(\varphi) - d$$

the defect of  $\varphi$ .

Note that by (i) in the introduction,  $d(\varphi) \geq 0$  and  $d(\varphi) = 0$  if and only if  $d = a(\varphi) + \text{ht}(\varphi)$ . If  $G$  is  $p$ -solvable or  $B$  has a cyclic defect group, then  $d(\varphi) = 0$  by Lemma 2.2, resp. ([8], Proposition 2.9) together with [13].

For  $x \in G$ , we denote by  $\tilde{x}$  the sum of all elements conjugate to  $x$  in  $G$ . Recall that for  $\chi \in \text{Irr}(G)$ , we have

$$\omega_\chi(\tilde{x}) = \frac{|G : C_G(x)|\chi(x)}{\chi(1)} \in R$$

for all  $x \in G$  (see for instance ([10], Chap. III, Theorem 2.25)). If we replace  $\chi$  by  $\varphi \in \text{IBr}_p(G)$ , then in general

$$\omega_\varphi(\tilde{x}) = \frac{|G : C_G(x)|\varphi(x)}{\varphi(1)}$$

need not to be in  $R$  for  $x \in G_{p'}$  as [15] shows. Instead we have

**Theorem 4.2.** *If  $\varphi \in \text{IBr}_p(B)$  where  $B$  is a  $p$ -block, then*

$$p^{d(\varphi)}\omega_\varphi(\tilde{x}) \in R$$

for all  $x \in G_{p'}$  and  $p^{d(\varphi)}\omega_\varphi(\tilde{x}) \in R^*$  for at least one  $x \in G_{p'}$ .

*Proof.* We have

$$\begin{aligned} p^{d(\varphi)}\omega_\varphi(\tilde{x}) &= \frac{p^{d(\varphi)}|G:C_G(x)|\varphi(x)}{\varphi(1)} \\ &= \frac{p^{a(\varphi)+\text{ht}(\varphi)-d}|G|\varphi(x)}{p^{a-d+\text{ht}(\varphi)}\varphi(1)_{p'}|C_G(x)|} \\ &= \frac{|G|_{p'}}{\varphi(1)_{p'}} \cdot \frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|}. \end{aligned}$$

Thus the assertion follows by Theorem 3.1. □

As a consequence of Theorem 4.2 we get an early result of Okuyama.

**Lemma 4.3.** ([12], Lemma 2.1) *If  $\varphi \in \text{IBr}_p(G)$  is of height 0, then  $\omega_\varphi(\tilde{x}) \in R$  for all  $x \in G_{p'}$ .*

*Proof.* The condition  $\text{ht}(\varphi) = 0$  implies  $a(\varphi) = d$  (see ([8], Theorem 2.1)) where  $d$  is the defect of the block to which  $\varphi$  belongs. Thus  $d(\varphi) = a(\varphi) + \text{ht}(\varphi) - d = 0$  and Theorem 4.2 leads to the assertion. □

**Remark 4.4.** a) In general  $p^{d(\varphi)}\omega_\varphi(\tilde{x}) \not\equiv p^{d(\psi)}\omega_\psi(\tilde{x}) \pmod{\pi R}$  if  $\varphi$  and  $\psi$  are irreducible Brauer characters in the same  $p$ -block: The principal 2-block of  $G = J_1$  has an irreducible Brauer character  $\varphi$  of degree  $56 = 7 \cdot 2^3$  with  $a(\varphi) = 1$  and  $\text{ht}(\varphi) = 3$ , hence  $d(\varphi) = 1$ . But the trivial Brauer character  $1_G$  has defect  $d(1_G) = 0$ . Thus  $p^{d(\varphi)}\omega_\varphi(\tilde{1}) = 2$  and  $p^{d(1_G)}\omega_{1_G}(\tilde{1}) = 1$ .

b) Suppose that  $d(\varphi) = 0$  for  $\varphi \in \text{IBr}_p(G)$ . Thus

$$\lambda_\varphi(\tilde{x}) = \omega_\varphi(\tilde{x}) + \pi R = k$$

for all  $x \in G_{p'}$ . In general  $\lambda_\varphi$  does not define the central character of  $kG$  with respect to the  $p$ -block to which  $\varphi$  belongs. For instance, one may take  $G = A_{14}$  and  $p = 2$ . According to GAP [4] the principal 2-block of  $G$  contains 15 irreducible Brauer characters including  $\varphi_1$  and  $\varphi_{21}$  (in notation of GAP these are X1 and X21) with

$$(a(\varphi_i), \text{ht}(\varphi_i)) = (10, 0), (3, 7)$$

for  $i = 1, 21$ , hence  $d(\varphi_i) = 0$ . However the functions  $\lambda_{\varphi_i}$  are different. This answers Feit's question (I) in ([3], Chap. IV, section 5, page 166) in the negative.

## 5. NORMAL SUBGROUPS

By ([3], Chap. III, Corollary 4.13) we know that a normal  $p$ -subgroup  $N$  of  $G$  is contained in the vertex of any irreducible Brauer character  $\varphi$  of  $G$ . Moreover  $|vx(\varphi)| = |N||vx(\bar{\varphi})|$  where  $\bar{\varphi}$  is the Brauer character of the module affording  $\varphi$  but regarded as a module of  $\bar{G} = G/N$ . Note that  $N$  is contained in the kernel of  $\varphi$ .

**Proposition 5.1.** *Let  $N$  be a normal subgroup of  $G$  with  $|N|_p = p^n$ ,  $|G|_p = p^a$  and  $\bar{G} = G/N$ . Suppose that  $\varphi \in \text{IBr}_p(G)$ . Then we have the following.*

- a)  $a(\varphi) \leq a(\bar{\varphi}) + n$ .
- b)  $a(\bar{\varphi}) + n - \text{ht}(\bar{\varphi}) \leq a(\varphi)$ .
- c) If  $d(\bar{\varphi}) = 0$ , then  $a(\varphi) = a(\bar{\varphi}) + n$ .

*Proof.* a) We prove that

$$\check{\varphi}(g) = \begin{cases} p^{a(\bar{\varphi})+n}\varphi(g), & \text{if } g \text{ is a } p\text{'-element,} \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character of  $G$ , so that  $a(\varphi) \leq a(\bar{\varphi}) + n$  because of the minimality of  $a(\varphi)$ . According to Brauer's characterization of generalized characters ([3], Chap. IV, Theorem 1.1), it suffices to prove that  $\check{\varphi}_E$  is a generalized character for any elementary subgroup  $E$  of  $G$ . Let

$E = P \times H$ , where  $P$  and  $H$  are  $p$ - and  $p'$ -subgroups of  $G$  respectively. Let  $\xi \in \text{Irr}(P)$  and  $\eta \in \text{Irr}(H)$ . We have

$$\begin{aligned} (\check{\varphi}_E, \xi \times \eta)_E &= \frac{1}{|E|} \sum_{y \in H} \sum_{x \in P} \check{\varphi}(xy) \overline{\xi(x) \eta(y)} \\ &= \xi(1) \cdot \frac{p^{a(\bar{\varphi})+n}}{|E|} \sum_{y \in H} \varphi(y) \overline{\eta(y)}. \end{aligned}$$

However, we know that the inflation  $\text{Inf}(\tilde{\varphi})$  of

$$\tilde{\varphi}(g) = \begin{cases} p^{a(\bar{\varphi})} \bar{\varphi}(\bar{g}), & \text{if } \bar{g} \text{ is a } p'\text{-element of } \bar{G}, \\ 0, & \text{otherwise} \end{cases}$$

is a generalized character of  $G$ . Computing  $((\text{Inf}(\tilde{\varphi}))_E, 1_P \times \eta)_E$ , we have

$$\begin{aligned} ((\text{Inf}(\tilde{\varphi}))_E, 1_P \times \eta)_E &= \frac{1}{|E|} \sum_{x \in P, y \in H} (\text{Inf}(\tilde{\varphi}))(xy) \overline{1_P(x) \eta(y)} \\ &= \frac{1}{|E|} \sum_{x \in P \cap N, y \in H} (\text{Inf}(\tilde{\varphi}))(xy) \overline{\eta(y)} \\ &= \frac{p^{a(\bar{\varphi})|P \cap N|}}{|E|} \sum_{y \in H} \varphi(y) \overline{\eta(y)}, \end{aligned}$$

and so  $\frac{p^{a(\bar{\varphi})+n}}{|E|} \sum_{y \in H} \varphi(y) \overline{\eta(y)}$  is an integer. Thus  $\check{\varphi}_E$  is a generalized character of  $E$ , as desired.

b), c) Let  $B$  be the block of defect  $d$  to which  $\varphi$  belongs and let  $\bar{B}$  be the block of defect  $\bar{d}$  to which  $\bar{\varphi}$  belongs. Note that

$$p^{a-d+\text{ht}(\varphi)} = \varphi(1)_p = \bar{\varphi}(1)_p = p^{\bar{a}-\bar{d}+\text{ht}(\bar{\varphi})}. \quad (2)$$

By ([11], Theorem 9.9 (a)), we have

$$d - \bar{d} = n + m \quad (3)$$

for some  $m \geq 0$ . Thus equation (2) implies that

$$\text{ht}(\varphi) = \text{ht}(\bar{\varphi}) + m. \quad (4)$$

Since  $p^{a(\varphi)}\varphi$  is quasi-projective we get

$$p^a \mid p^{a(\varphi)}\varphi(1)_p = p^{a(\varphi)+a-d+\text{ht}(\varphi)},$$

hence

$$a(\varphi) \geq d - \text{ht}(\varphi).$$

By (3) and (4), it follows

$$a(\varphi) \geq \bar{d} + n + m - (\text{ht}(\bar{\varphi}) + m) \geq a(\bar{\varphi}) + n - \text{ht}(\bar{\varphi})$$

and in case  $d(\bar{\varphi}) = 0$ , i.e.,  $a(\bar{\varphi}) + \text{ht}(\bar{\varphi}) = \bar{d}$ , we obtain

$$a(\varphi) \geq \bar{d} + n + m - (\text{ht}(\bar{\varphi}) + m) = a(\bar{\varphi}) + n.$$

□

Note that the left hand side of the inequality of Proposition 5.1 b) might be negative.

**Theorem 5.2.** *Let  $N$  be a normal  $p$ -subgroup of  $G$  of order  $p^n$ , let  $\bar{G} = G/N$  and let  $\varphi \in \text{IBr}_p(G)$ . Then the following are equivalent.*

- a)  $a(\varphi) = a(\bar{\varphi}) + n$ .
- b) For  $x \in G_{p'}$  we have

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^* \iff x \in C_G(N) \text{ and } \frac{p^{a(\bar{\varphi})}\bar{\varphi}(\bar{x})}{|C_{\bar{G}}(\bar{x})|} \in R^*.$$

*Proof.* a)  $\implies$  b) Let  $x \in G_{p'}$  such that  $\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^*$ . Thus, if  $\nu$  denotes the valuation on  $K$  with  $\nu(p) = 1$ , then

$$a(\varphi) + \nu(\varphi(x)) = \nu(|C_G(x)|).$$

Since

$$C_G(x)/C_N(x) \cong C_G(x)N/N \leq C_{\bar{G}}(\bar{x})$$

we get

$$|C_G(x)| \leq |C_{\bar{G}}(\bar{x})||C_N(x)|.$$

It follows that

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= \nu(|C_G(x)|) \\ &\stackrel{(i)}{\leq} \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|C_N(x)|) \\ &\stackrel{(ii)}{\leq} a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n. \end{aligned}$$

Since  $\nu(\varphi(x)) = \nu(\bar{\varphi}(\bar{x}))$  and by assumption  $a(\varphi) = a(\bar{\varphi}) + n$ , we have equality in (i) and (ii). Hence

$$(i') \quad C_N(x) = N$$

and

$$(ii') \quad a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) = \nu(|C_{\bar{G}}(\bar{x})|).$$

Note that (ii') means that  $\frac{p^{a(\bar{\varphi})}\bar{\varphi}(\bar{x})}{|C_{\bar{G}}(\bar{x})|} \in R^*$ .

Vice versa, if (i') and (ii') holds then

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n \\ &= \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|C_N(x)|) \\ &= \nu(|C_G(x)|) \\ &\leq a(\varphi) + \nu(\varphi(x)), \end{aligned}$$

hence

$$a(\varphi) + \nu(\varphi(x)) = \nu(|C_G(x)|)$$

which shows that

$$\frac{p^{a(\varphi)}\varphi(x)}{|C_G(x)|} \in R^*.$$

b)  $\implies$  a) We have

$$\begin{aligned} a(\varphi) + \nu(\varphi(x)) &= \nu(|C_G(x)|) \\ &= \nu(|C_{\bar{G}}(\bar{x})|) + \nu(|N|) \\ &= a(\bar{\varphi}) + \nu(\bar{\varphi}(\bar{x})) + n. \end{aligned}$$

Thus  $a(\varphi) = a(\bar{\varphi}) + n$ .  $\square$

**Proposition 5.3.** *Let  $N$  be a normal  $p$ -subgroup of  $G$  and let  $\bar{G} = G/N$ . If  $\varphi \in \text{IBr}_p(G)$ , then  $a(\bar{\varphi}) \leq a(\varphi)$ .*

*Proof.* By induction we may assume that  $N$  is abelian. Due to the Alperin-Collins-Sibley Theorem ([7], Chap. II, Theorem 11.14) we have

$$\Phi_\varphi(x) = \rho(x)\Phi_{\bar{\varphi}}(\bar{x})$$

for all  $x \in G_{p'}$  where  $\rho$  is the Brauer character afforded by the conjugation action of  $G$  on  $kN$ . Since  $N$  acts trivially on  $kN$  we obtain  $\rho(x) = \rho(\bar{x})$ . Thus we have

$$p^{a(\varphi)}\varphi(x) = \sum_{\psi \in \text{IBr}_p(G)} a_\psi \Phi_\psi(x) = \sum_{\bar{\psi} \in \text{IBr}_p(\bar{G})} a_{\bar{\psi}} \rho(\bar{x}) \Phi_{\bar{\psi}}(\bar{x}).$$

Note that  $\rho \cdot \Phi_{\bar{\psi}}$  is a sum of projective characters of  $\bar{G}$  ([11], Lemma 2.25), hence

$$\rho(\bar{x})\Phi_{\bar{\psi}}(\bar{x}) = \sum_{\bar{\lambda} \in \text{IBr}_p(\bar{G})} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x}).$$

Therefore we obtain

$$p^{a(\varphi)}\varphi(x) = \sum_{\bar{\psi} \in \text{IBr}_p(\bar{G})} \sum_{\bar{\lambda} \in \text{IBr}_p(\bar{G})} a_{\bar{\psi}} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x}).$$

Since  $N$  is in the kernel of  $\varphi$  we finally have

$$p^{a(\varphi)}\bar{\varphi}(\bar{x}) = p^{a(\varphi)}\varphi(x) = \sum_{\bar{\psi} \in \text{IBr}_p(\bar{G})} \sum_{\bar{\lambda} \in \text{IBr}_p(\bar{G})} a_{\bar{\psi}} b_{\bar{\lambda}} \Phi_{\bar{\lambda}}(\bar{x})$$

with  $a_{\bar{\psi}} b_{\bar{\lambda}} \in \mathbb{Z}$ . This implies  $a(\bar{\varphi}) \leq a(\varphi)$ .  $\square$

**Example 5.4.** Let  $G = 3^6 : 2M_{12}$  where  $M_{12}$  is the Mathieu group on 12 letters and let  $p = 3$ . Then  $G$  consists only of the principal 3-block and for  $a(\varphi)$  where  $\varphi \in \text{IBr}_3(G)$  one computes

$$9, 9, 9, 9, 8, 8, 7, 7, 9, 9, 9, 9, 8, 8, 8, 8, 7, 7, 6.$$

The factor group  $\bar{G} = 2M_{12}$  has exactly four blocks and we compute for  $a(\bar{\varphi})$

$$\begin{array}{ll} \text{block 1:} & 3, 3, 3, 3, 2, 2, 1, 1 & \text{block 2:} & 3, 3, 3, 3, 2, 2, 2, 2 \\ \text{block 3:} & 1, 1 & \text{block 4:} & 0 \end{array}$$

Thus we get  $a(\varphi) = a(\bar{\varphi}) + 6$  for all  $\varphi$ .

Many other examples lead to the following conjecture.

**Conjecture 5.5.** If  $N$  is a normal  $p$ -subgroup of  $G$  of order  $p^n$ , then  $a(\varphi) = a(\bar{\varphi}) + n$  for all  $\varphi \in \text{IBr}_p(G)$ .

The conjecture holds true for  $p$ -solvable groups, by Lemma 2.2 and Proposition 5.1 part c).

## 6. THE GREEN CORRESPONDENCE

Let  $\mathbb{Z}[\text{IBr}_p(G)] = \{\sum_{\varphi \in \text{IBr}_p(G)} a_\varphi \varphi \mid a_\varphi \in \mathbb{Z}\}$  denote the ring of  $\mathbb{Z}$ -linear combinations of irreducible Brauer characters. As for irreducible Brauer characters we may define Hilbert divisors  $p^{a(\psi)}$  for any  $\psi \in \mathbb{Z}[\text{IBr}_p(G)]$  with  $a(\psi) \leq d$  in case  $\psi \in \mathbb{Z}[\text{IBr}_p(B)]$  where  $B$  is a  $p$ -block of defect  $d$  (see the proof of Theorem 2.1 in [8]).

**Lemma 6.1.** *Let  $H \leq G$ .*

- a) *If  $\psi = \sum_{\varphi \in \text{IBr}_p(G)} a_\varphi \varphi \in \mathbb{Z}[\text{IBr}_p(G)]$ , then  $a(\psi) \leq \max\{a(\varphi) \mid a_\varphi \neq 0\}$  and  $a(\psi|_H) \leq a(\psi)$ .*
- b) *If  $\lambda \in \mathbb{Z}[\text{IBr}_p(H)]$ , then  $a(\lambda^G) \leq a(\lambda)$ .*

*Proof.* This is a consequence of the fact that induction and restriction of projective modules are projective.  $\square$

**Proposition 6.2.** *Let  $M$  be an indecomposable  $kG$ -module with Brauer character  $\varphi$ . Then  $p^{a(\varphi)} \leq |vx(\varphi)|$ .*

*Proof.* The proof is exactly the same as for simple modules (see Proposition 2.6 in [8]).  $\square$

**Theorem 6.3.** *Let  $M$  be an indecomposable  $kG$ -module with Brauer character  $\varphi$  and  $|vx(\varphi)| = p^v$ . Let  $f(\varphi)$  denote the Brauer character of the Green correspondent of  $M$  in  $(G, V, H = N_G(V))$ . Then  $a(\varphi) = v$  if and only if  $a(f(\varphi)) = v$ .*

*Proof.* We may assume that  $v > 1$ . First we consider the case that  $a(\varphi) = v$ . According to the Green correspondence ([7], Chap. II, Theorem 4.1) or ([10], Chap. 4, Theorem 4.3 b)) we have

$$f(M)^G = M \oplus o(\mathcal{X})$$

where the indecomposable direct summands in  $o(\mathcal{X})$  have vertices in

$$\mathcal{X} = \{W \leq G \mid W \leq V \cap V^g, g \in G \setminus H\}.$$

Thus for Brauer characters we get the equation

$$\beta^G = \varphi + \psi$$

where  $\beta$  and  $\psi$  are the Brauer characters of  $f(M)$  and  $o(\mathcal{X})$  respectively. Multiplying this equation by  $p^{v-1}$  we get

$$(p^{v-1}\beta)^G = p^{v-1}\varphi + p^{v-1}\psi.$$

Suppose that  $a(\beta) \leq v - 1$ . Then  $(p^{v-1}\beta)^G$  is quasi-projective. Since all vertices occurring in the direct summands of  $o(\mathcal{X})$  have order less than or equal to  $v - 1$  we get, by Proposition 6.2, that  $p^{v-1}\psi$  is also quasi-projective. Thus  $p^{v-1}\varphi$  is quasi-projective, a contradiction.

Now let  $a(f(\varphi)) = a(\beta) = v$ . By ([10], Chap. 4, Lemma 4.2 (i)) we have

$$(f(M)^G)|_H = f(M) \oplus o(\mathcal{Y})$$

where the indecomposable direct summands in  $o(\mathcal{Y})$  have vertices in

$$\mathcal{Y} = \{W \leq G \mid W \leq V^g \cap H, g \in G \setminus H\}.$$

Thus in terms of Brauer characters we

$$\beta^G|_H = \beta + \eta$$

where  $\eta$  is the Brauer character of  $o(\mathcal{Y})$ . Applying ([10], Chap. 4, Lemma 4.2 (ii) (a)) we see that all indecomposable components of  $\mathcal{Y}$  have a vertex of order less than  $p^v$ . Now we assume that  $a(M) = a(\varphi) \leq v - 1$ . Since  $\beta^G = \varphi + \psi$  we get

$$a(\beta^G|_H) \leq a(\beta^G) \leq \max\{a(\varphi), a(\psi)\} \leq v - 1$$

and the equation  $\beta^G|_H = \beta + \eta$  leads to  $a(f(M)) = a(\beta) \leq v - 1$ , a contradiction.  $\square$

**Corollary 6.4.** *Let  $G$  be a  $p$ -solvable group and let  $f(\varphi)$  denote the Green correspondent of  $\varphi \in \text{IBr}_p(G)$  in  $(G, vx(\varphi), N_G(vx(\varphi)))$ . Then  $a(\varphi) = a(f(\varphi))$ .*

*Proof.* By ([8], Proposition 2.7), we have  $p^{a(\varphi)} = |vx(\varphi)|$  and we may apply Theorem 6.3.  $\square$

**Example 6.5.** a) Let  $G = A_5$  be the alternating group on 5 letters and  $p = 2$ . Then one easily computes  $a(\varphi) = a(f(\varphi))$  for all  $\varphi \in \text{IBr}_2(G)$  where  $f(\varphi)$  is the Brauer character of the Green correspondent of the module affording  $\varphi$  in  $(G, vx(\varphi), N_G(vx(\varphi)))$ .

b) Let  $G = M_{12}$  be the Mathieu group on 12 letters and let  $p = 2$ . We have  $|G|_2 = 2^6$ . Again  $a(\varphi) = a(f(\varphi)) = 6, 5, 4$  for  $\varphi \in \text{IBr}_2(B_0)$  of dimension 1, 10, 44 in the principal 2-block  $B_0$ . For the other block  $B_2$  with  $d = 2$  we have  $a(\varphi) = a(f(\varphi)) = 2$  for all  $\varphi \in \text{IBr}_2(B_2)$ . The computations are based on the information given in [14].

c) The group  $G = M_{22}$  has only one 2-block. The degrees of the irreducible Brauer characters are 1, 10, 10, 34, 70, 70, 98 and all vertices are equal to the Sylow 2-subgroup of  $G$  which is of order  $2^7$ . For  $a(\varphi)$  one computes 7, 6, 6, 6, 6, 6, 6 and for  $a(f(\varphi))$  as well 7, 6, 6, 6, 6, 6, 6. The computations are again based on the information given in [14].

d) Let  $G = M_{24}$  and  $p = 3$ . The principal 3-block  $B_0$  has irreducible Brauer characters of degrees 1, 22, 231, 483, 770, 770, 1243. Apart from the character of degree 483 which has a vertex of order 9, all have a Sylow 3-subgroup (of order 27) as vertex. The exponents  $a(\varphi)$  are 3, 3, 2, 2, 3, 3, 3. With the information of [1] we again have  $a(\varphi) = a(f(\varphi))$  for all  $\varphi \in \text{IBr}_3(B_0)$ . The information about the Green correspondents may be taken from [1].

## 7. $p$ -SOLVABLE GROUPS

It is well-known that the Cartan numbers of a  $p$ -block  $B$  of defect  $d$  of a  $p$ -solvable group are bounded by  $p^d$  (see for instance ([3], Chap. X, Theorem 4.2) or [6]). Actually a stronger result holds true if  $B$  contains at least two irreducible Brauer characters.

**Proposition 7.1.** *Let  $B$  be a  $p$ -block of the  $p$ -solvable group of defect  $d$  and let  $C = (c_{\varphi\psi})$  denote the Cartan matrix of  $B$ . If  $l(B) \geq 2$ , then  $c_{\varphi\psi} < p^d$  for all  $\varphi, \psi \in \text{IBr}_p(B)$ .*

*Proof.* Clearly,  $d > 0$  since  $B$  contains at least two irreducible Brauer characters. Suppose that  $c_{\varphi\psi} = p^d$  for some  $\varphi, \psi \in \text{IBr}_p(B)$ . As

$$p^d = c_{\varphi\psi} \leq \max\{|vx(\varphi)|, |vx(\psi)|\} \leq p^d$$

by [6], we may assume that  $p^d = |vx(\varphi)| \geq |vx(\psi)|$ . According to [3, Chap. X, Theorem 1.8] we also have

$$\Phi_\beta(1) = |vx(\beta)|\beta(1)$$

for all  $\beta \in \text{IBr}_p(B)$ . Since

$$\Phi_\varphi(1) = c_{\varphi\varphi}\varphi(1) + c_{\varphi\psi}\psi(1) + \dots \geq p^d\psi(1) \geq |vx(\psi)|\psi(1) = \Phi_\psi(1) \quad (5)$$

we get

$$\Phi_\varphi(1) \geq \Phi_\psi(1).$$

On the other hand

$$\Phi_\psi(1) = c_{\psi\psi}\psi(1) + c_{\varphi\psi}\varphi(1) + \dots \geq p^d\varphi(1) = \Phi_\varphi(1)$$

leads to  $\Phi_\psi = \Phi_\varphi$  and hence  $\varphi(1) \geq \psi(1)$  since  $p^d = |vx(\varphi)| \geq |vx(\psi)|$ . From this we see that in equation (5) everywhere holds equality which implies  $c_{\varphi\varphi} = 0$ , a contradiction.  $\square$

**Proposition 7.2.** *Let  $G$  be a  $p$ -solvable group and let  $B$  be a  $p$ -block of defect  $d$ . If the defect group of  $B$  is abelian, then  $a(\varphi) = d$  for all  $\varphi \in \text{IBr}_p(B)$ .*

*Proof.* By Brauer's height zero conjecture which has an affirmative answer for  $p$ -solvable groups due to [5], the heights of all irreducible complex characters in  $B$  are zero. Thus the heights of all irreducible Brauer characters in  $B$  are zero since they are liftable by the Fong-Swan Theorem. The assertion now follows by ([8], Theorem 2.1).  $\square$

**Example 7.3.** Let  $J_1$  denote the first simple Janko group. Recall that the Sylow 2-subgroup of  $G$  is elementary abelian of order 8. The principal 2-block of  $G$  contains 5 irreducible Brauer characters of degree 1, 20, 56, 56, 76. The heights are 0, 2, 3, 3, 2. For  $a(\varphi)$  one computes 3, 1, 1, 1, 1. Thus the above Proposition does not hold in general.

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