TRIVIAL INTERSECTION OF BLOCKS AND NILPOTENT SUBGROUPS

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Abstract. Let p, q be different primes and suppose that the principal p- and the principal q-block of a finite group have only one irreducible complex character in common, namely the trivial one. We conjecture that this condition implies the existence of a nilpotent Hall $\{p, q\}$-subgroup and prove that a minimal counter-example must be an almost simple group where pq divides the order of its simple nonabelian normal subgroup. As an immediate consequence we obtain that the conjecture holds true for p-solvable or q-solvable groups. Furthermore, we prove the conjecture in case $2 \in \{p, q\}$ using the classification theorem of finite simple groups. Finally, we consider the situation that the intersection of an arbitrary p-block with an arbitrary q-block contains only one irreducible character.

1. Introduction

For a finite group G and a p-block $B(G)_p$ of G we denote by $\mathrm{Irr}(G)$ resp. $\mathrm{Irr}(B(G)_p)$ the set of irreducible complex characters of G, resp. the set of those which belong to $B(G)_p$, where p is a prime. Furthermore, we use the notation 1_G for the trivial character of G and $B_0(G)_p$ for the principal p-block of G. Throughout the paper let q be a prime different from the prime p. Inspired by Brauer's problem [8], many papers in the literature deal with relations between p- and q-blocks of G and properties of the group structure. In this sense the following two theorems have been proved in ([6], Theorem 4.1 and Proposition 4.2).

Theorem 1.1. (Bessenrodt-Zhang) *We have* $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ *for all primes* $p \neq q$ *dividing* $|G|$ *if and only if* G *is nilpotent.*

What they really proved is the following from which Theorem 1.1 directly follows.

Theorem 1.2. (Bessenrodt-Zhang) *Let* p *be a fixed prime dividing* $|G|$. *Then we have* $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ *for all primes* $q \neq p$ *if and only if* $G = P \times \mathrm{O}_{p'}(G)$ *where* P *is a Sylow* p*-subgroup of* G.

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We conjecture here a more general fact which obviously implies Theorem 1.1, namely

Conjecture 1.3. *If* $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$ *for a pair of primes* p, q*, then* G *has a nilpotent Hall* $\{p, q\}$*-subgroup.*

Note that the converse of Conjecture 1.3 is false. We may take for G a dihedral group of order $|G| = 2pq$ where $2, p, q$ are pairwise different primes. The cyclic subgroup of order pq is a Hall $\{p, q\}$-subgroup. Furthermore, the linear nontrivial character of G belongs to the principal p- and q-block.

In case G is a simple group Conjecture 1.3 has an affirmative answer, by ([10], Theorem 1.2). The only examples satisfying the assumption of Conjecture 1.3 are J_1 for $\{p, q\} = \{3, 5\}$ and J_4 for $\{p, q\} = \{5, 7\}$. Both groups have a nilpotent Hall $\{p, q\}$-subgroup.

We would like to mention here that in [27] and [4] the existence of nilpotent Hall subgroups is described by pure group theoretical properties. In addition, recently, a conjectural characterization of the property in terms of character degrees and principal blocks is put forward by G. Malle and G. Navarro [25].

In this note we reduce Conjecture 1.3 to almost simple groups and prove it in special cases.

Theorem 1.4. *If* G *is a minimal counter-example to Conjecture 1.3 w.r.t. the order, then* $S \leq G \leq \mathrm{Aut}(S)$ *where* S *is a nonabelian simple group with* $pq \mid |S|$*. Moreover,* $G = SC_G(P) = SC_G(Q)$ *where* P *is a Sylow* p*-subgroup and* Q *is a Sylow* q*-subgroup of* S*.*

As a consequence Conjecture 1.3 obviously holds true for p- or q-solvable groups as well. Using extensively the character theory of groups of Lie type we can answer Conjecture 1.3 if one of the primes is even.

Theorem 1.5. *Conjecture 1.3 holds true if* $2 \in \{p, q\}$*.*

However, the proof of Theorem 1.5 requires the following stronger version of Theorem 1.4. The added improvement relies on the classification of finite simple groups which is not needed for Theorem 1.4.

Theorem 1.6. *If* G *is a minimal counter-example to Conjecture 1.3 w.r.t. the order, then* $S < G \leq \mathrm{Aut}(S)$ *where* S *is a simple nonabelian group with* $pq \mid |S|$ *and* $\gcd(|G/S|, pq) = 1$*. Moreover,* $G = S\,\mathrm{O}_{p'}(C_G(P)) = S\,\mathrm{O}_{q'}(C_G(Q))$ *where* P *is a Sylow* p*-subgroup and* Q *is a Sylow* q*-subgroup of* S*.*

Instead of considering the intersection of principal blocks we may look at groups with the following property:

$$|\mathrm{Irr}(B(G)_p) \cap \mathrm{Irr}(B(G)_q)| = 1$$

where $B(G)_p$ and $B(G)_q$ are arbitrary blocks for the prime p, resp. q. If this condition holds, then one is tempted to ask whether a fixed defect group of $B(G)_p$ must centralize a suitable defect group of $B(G)_q$. Unfortunately, this is not always the case as the next example shows.

Example 1.7. Let $G = Co1$ be the first Conway simple group. Then G has a 2-block $B(G)_2$ with defect group $\langle 2a, 4e\rangle$ in the notation of the ATLAS [12]. There is also a 13-block $B(G)_{13}$ of defect 1 which intersects $B(G)_2$ only in one irreducible character. However, a defect group of $B(G)_2$ never commutes with a defect group of $B(G)_{13}$ since G has no element of order $4 * 13$.

Theorem 1.8. *Let $B(G)_p$ and $B(G)_q$ be blocks of maximal defect and suppose that the Sylow p- and Sylow q-subgroups of G are cyclic. If $|\mathrm{Irr}(B(G)_p) \cap \mathrm{Irr}(B(G)_q)| = 1$, then G has a nilpotent Hall $\{p, q\}$-subgroup.*

Note that Theorem 1.8 answers Conjecture 1.3 in case that the Sylow p- and Sylow q-subgroups of G are cyclic. We may ask here the question whether Theorem 1.8 holds true if we drop the assumption that the blocks are of maximal defect. This means, does a defect group of $B(G)_p$ centralizes a suitable defect group of $B(G)_q$ if $|\mathrm{Irr}(B(G)_p)\cap\mathrm{Irr}(B(G)_q)| = 1$ for blocks $B(G)_p$ and $B(G)_q$?

Finally we prove this for a special case in the last section .

Theorem 1.9. *Let $B(G)_p$ and $B(G)_q$ be blocks of the symmetric group S_n. If $|\mathrm{Irr}(B(G)_p)\cap \mathrm{Irr}(B(G)_q)| = 1$, then a defect group of $B(G)_p$ centralizes a suitable defect group of $B(G)_q$.*

2. Reduction to almost simple groups

In this section we reduce Conjecture 1.3 to almost simple groups. If $f_{0,p}(G) = \sum_{g\in G} f_g g$ denotes the block idempotent of the principal p-block $B_0(G)_p$ over a splitting field of characteristic p for a finite group G, then $\mathrm{O}_{f_{0,p}}(G)$ is defined as the subgroup of G generated by

$$\mathrm{supp}(f_{0,p}(G)) = \{g \mid f_g \neq 0\}.$$

As usual, the notations $E(G), F(G)$ and $F^*(G)$ denote the layer, the Fitting subgroup and the generalized Fitting subgroup of G, respectively, see [19, 20].

Lemma 2.1. ([32], Lemma 1.3) *Let N be a normal subgroup of a finite group G. Suppose that $C_G(P) \leq N$ for a Sylow p-subgroup P of N. Then $f_{0,p}(N) = f_{0,p}(G)$.*

Lemma 2.2. *Let N be a normal subgroup of a finite group G. Then the following statements are equivalent:*

1. $f_{0,p}(N) = f_{0,p}(G)$.
2. $\mathrm{O}_{f_{0,p}}(G) \subseteq N$, *i.e., $f_{0,p}(G)$ is supported on elements of N.*
3. *The principal p-block of G is the unique p-block of G covering the principal p-block of N.*
4. *The irreducible characters of G/N all lie in the principal p-block of G.*

Proof. The equivalence of (1), (2), and (3) is immediate from the theory of covering blocks and the equivalence of (3) and (4) is a consequence of block domination by ([28], Ch. V, Lemmas 5.6 and 8.6). □

Lemma 2.3. *Let p and q be primes, and let G be a finite group. Suppose that pq divides $|G|$, and $\mathrm{Irr}(B_0(G))_p \cap \mathrm{Irr}(B_0(G))_q = \{1_G\}$. Then for any normal subgroup N of G such that $f_{0,p}(N) = f_{0,p}(G)$, the following statements hold:*

(1) *q does not divide $|G/N|$.*
(2) *$\mathrm{Irr}(B_0(N))_p \cap \mathrm{Irr}(B_0(N))_q = \{1_N\}$.*

Proof. (1) Let $\bar{G} = G/N$. By Lemma 2.2 (1) and (4), we have $\mathrm{Irr}(\bar{G}) \subseteq \mathrm{Irr}(B_0(G)_p)$. Obviously, $\mathrm{Irr}(B_0(\bar{G})_q) \subseteq \mathrm{Irr}(B_0(G)_q)$. Thus, by assumption

$$\mathrm{Irr}(\bar{G}) \cap \mathrm{Irr}(B_0(\bar{G})_q) = \{1_G\}.$$

This implies that $\bar{G}$ is a q'-group.
(2) By (1), we know that $q \mid |N|$. We may assume $p \mid |N|$, since otherwise the conclusion trivially holds. Now the assertion of (2) follows by Lemma 2.2 (1) and (3) and ([10], Lemma 2.6) applied with the role of p by q. □

Lemma 2.4. *Suppose that G is a minimal counter-example to Conjecture 1.3. Suppose that N is a proper normal subgroup of G and let P be a Sylow p-subgroup of N and Q a Sylow q-subgroup of N. Then $f_{0,p}(G) \neq f_{0,p}(N)$ and $G = NC_G(P) = NC_G(Q)$.*

Proof. Suppose that $f_{0,p}(G) = f_{0,p}(N)$ for some proper normal subgroup N of G. Without loss of generality, we may we assume that N is a maximal normal subgroup of G. By Lemma 2.3 (1), N contains a Sylow q-subgroup of G and by Lemma 2.3 (2) and the minimality of G as a counterexample, there exists a Sylow q-subgroup Q of N (hence of G) such that $C_N(Q)$ contains a Sylow p-subgroup of N.

If $f_{0,q}(G) = f_{0,q}(N)$, then by Lemma 2.3 (2), applied with the roles of p and q reversed, N also contains every Sylow p-subgroup of G. We obtain a contradiction. So we may assume that $f_{0,q}(G) \neq f_{0,q}(N)$. Note that $G = NN_G(Q)$ by the Frattini argument. Furthermore $NC_G(Q)$ is a normal subgroup of G. If $C_G(Q) \subseteq N$ then since Q is a Sylow q-subgroup of N, it follows by Lemma 2.1 that $f_{0,q}(G) = f_{0,q}(N)$, a contradiction. Therefore, $C_G(Q) \not\subseteq N$, and so by the maximality of N, we have $G = NC_G(Q)$. In particular, $|G|_p|C_N(Q)|_p = |N|_p|C_G(Q)|_p$. Now the previous paragraph shows that $|C_G(Q)|_p = |G|_p$, and again we obtain a contradiction. This proves the first assertion.

For the second assertion, let P_0 be a Sylow p-subgroup of $H := NC_G(P)$ containing P. Then $C_G(P_0) \leq C_G(P) \leq H$ and by the Frattini argument H is normal in G. So, by Lemma 2.1, $f_{0,p}(G) = f_{0,p}(H)$. Hence by the previous assertion $H = G$. □

Proof of Theorem 1.4

Let G be a minimal counter-example with respect to $|G|$. Thus $pq \mid |G|$.

(1) *G has a unique minimal normal subgroup N.*

Proof. Suppose that G has two minimal normal subgroups M and N. If $p \mid |G/N|$ but $q \nmid |G/N|$, then any Sylow p-subgroup of G/N is a nilpotent Hall $\{p,q\}$-subgroup of G/N, while if $pq \mid |G/N|$ then G/N has a nilpotent Hall $\{p,q\}$-subgroup by the fact that $\mathrm{Irr}(B_0(G/N)_p) \cap \mathrm{Irr}(B_0(G/N)_q) = \{1_{G/N}\}$ and the minimality of G. Hence $\widehat{G} := G/N \times G/M$ has a nilpotent Hall $\{p,q\}$-subgroup in any case. By a theorem of Wielandt [31], all of the Hall $\{p,q\}$-subgroups of $\widehat{G}$ are conjugate and each $\{p,q\}$-subgroup of $\widehat{G}$ is contained in some Hall $\{p,q\}$-subgroup of $\widehat{G}$. Since G can be viewed as a subgroup of $\widehat{G}$, it follows that G also has a nilpotent Hall $\{p,q\}$-subgroup, a contradiction. □

(2) $F(G) = 1$.

Proof. Suppose $U := O_p(G) \neq 1$. By Lemma 2.4 applied with $N = U$, $G = UC_G(U)$, whence $C_G(U)$ contains a Sylow q-subgroup of G. By the minimality of G as a counterexample G/U has a nilpotent Hall $\{p,q\}$-subgroup. So there exists a Sylow p-subgroup P of G and a Sylow q-subgroup Q of G such that Q centralizes P/U and U. According to ([3], Lemma A.2), Q centralizes P. This means that G has a nilpotent Hall $\{p,q\}$-subgroup, a contradiction. Similarly, $O_q(G) = 1$. So $|F(G)|$ is relatively prime to pq. Consequently, if $F(G) \neq 1$, then $G/F(G)$ and hence G has a nilpotent Hall $\{p,q\}$-subgroup, a contradiction. □

(3) $E(G)$ *is simple.*

Proof. By Steps (1) and (2), $E(G) =: N = S_1 \times \cdots \times S_t$ is a minimal normal subgroup of G, where $S_i \cong S$ is non-abelian simple. Suppose that $t \neq 1$. In particular, $N \neq G$. By the minimality of G as a counterexample, either p divides $|N|$ or q divides $|N|$. Without loss of generality we may assume $p \mid |N|$. Let P_i be a Sylow p-subgroup of S_i and let $P = P_1 \times \cdots \times P_t$. By Lemma 2.4, $G = NC_G(P)$. So for any $g \in G$, $P_1^g \leq S_1 \cap S_1^g$ and S_1^g is a normal subgroup of N. Hence $S_1 \cap S_1^g$ is a non-trivial normal subgroup of S_1, hence $S_1 \cap S_1^g = S_1$. Consequently, S_1 is normal in G, contradicting the minimality of N. It follows that N is non-abelian simple. □

(4) $S \leq G \leq \mathrm{Aut}(S)$ *and* $pq \mid |S|$. *Moreover,* $G = SC_G(P) = SC_G(Q)$ *where* P *is a Sylow* p*-subgroup and* Q *is a Sylow* q*-subgroup of* S*.*

Proof. By Steps (2) and (3), $F^*(G) = S$ is non-abelian simple, whence $S \leq G \leq \mathrm{Aut}(S)$. If $G \neq S$, then by Lemma 2.4, $G = SC_G(P) = SC_G(Q)$ where P is a Sylow p-subgroup and Q is a Sylow q-subgroup of S. Suppose that without loss of generality $p \mid |S|$. If $q \nmid |S|$ then $C_G(P)$ contains a Sylow q-subgroup Q of G. By the minimality of G, $G/S \cong PC_G(P)/PC_S(P)$ has a nilpotent Hall $\{p,q\}$-subgroup. Write P_1 for a Sylow p-subgroup of $PC_G(P)$ (and hence of G) such that Q centralizes P_1/P. Then Q centralizes P_1 by ([3], Lemma A.2), namely G has a nilpotent Hall $\{p,q\}$-subgroup, a contradiction. This completes the proof. □

□

3. Reduction to almost simple groups with generalized p'-core

In order to prove Theorem 1.6, we need the following definition which is due to Bender in [5].

Definition 3.1.
a) A finite group G is called a p^*-group if the following two conditions hold.
  (i) $\mathrm{O}^p(G) = G$, i.e., G does not have a nontrivial p-factor group.
  (ii) Whenever $N \trianglelefteq G$ and $P \in \mathrm{Syl}_p(N)$, then $G = NC_G(P)$.

b) $\mathrm{O}_{p^*}(G) = \langle N \mid N \trianglelefteq G, N \text{ is a } p^*\text{-group}\rangle$ is called the generalized p'-core of G.

As usual we define $\mathrm{O}_{p^*,p}(G)$ by

$$\mathrm{O}_{p^*,p}(G)/\mathrm{O}_{p^*}(G) = \mathrm{O}_p(G/\mathrm{O}_{p^*}(G)).$$

Theorem 3.2. ([20], Chap. X, Section 14) *The following holds true.*
  a) $\mathrm{O}_{p^*}(G)$ *is the largest normal p^*-subgroup of G.*
  b) *If $P \in \mathrm{Syl}_p(\mathrm{O}_{p^*,p}(G))$, then $C_G(P) \leq \mathrm{O}_{p^*,p}(G)$.*
  c) *G is p-constrained if and only if $\mathrm{O}_{p^*}(G) = \mathrm{O}_{p'}(G)$.*

It is well known that $\mathrm{O}_{f_{0,p}}(G) = \mathrm{O}_{p'}(G)$ if G is p-constrained ([20], Chap. VII, Theorem 13.6). In general, we have

Theorem 3.3. ([33], Theorem 2.1 and 2.6)

$$\mathrm{O}_{f_{0,p}}(G) \leq \mathrm{O}_{p^*}(G)$$

with equality if $p \neq 2$.

Lemma 3.4. *Suppose that $pq \mid |G|$. If $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$, then*
  (1) *$q \nmid |G/\mathrm{O}_{p*}(G)|$ and $p \nmid |G/\mathrm{O}_{q*}(G)|$.*
  (2) *$\mathrm{Irr}(B_0(N)_p) \cap \mathrm{Irr}(B_0(N)_q) = \{1_N\}$ for any $N \trianglelefteq G$ containing $\mathrm{O}_{p^*}(G)$.*

Proof. By Theorem 3.3, $f_{0,p}(G)$ is supported on elements of $\mathrm{O}_{p*}(G)$ and $f_{0,q}(G)$ is supported on elements of $\mathrm{O}_{q*}(G)$. Thus the assertions of part (1) and (2) follow from Lemma 2.2 (1) and (2) and Lemma 2.3. □

Corollary 3.5. *Suppose that $pq \mid |G|$.*
If $\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}$, then $G = \mathrm{O}_{p^*}(G)\,\mathrm{O}_{q*}(G)$.

Proof. Let $\bar{G} = G/\mathrm{O}_{p^*}(G)$. By Theorem 3.3, $f_{0,p}(G)$ is supported on elements of $\mathrm{O}_{p*}(G)$. So, by Lemma 2.2 (2) and (4), we have $\mathrm{Irr}(\bar{G}) \subseteq \mathrm{Irr}(B_0(G)_p)$. It follows that

$$\mathrm{Irr}(G/(\mathrm{O}_{p^*}(G)\,\mathrm{O}_{q^*}(G))) \subseteq \mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\}.$$

Thus $G = \mathrm{O}_{p^*}(G)\,\mathrm{O}_{q^*}(G)$. □

Lemma 3.6. *Let $M \trianglelefteq G$ with $|G/M| = p$ and suppose that G satisfies the assumption of Conjecture 1.3. If M has a nilpotent Hall $\{p, q\}$-subgroup, then G has a nilpotent Hall $\{p, q\}$-subgroup.*

Proof. Let $P_0 \times Q$ be a nilpotent Hall $\{p,q\}$-subgroup of M where $Q \in \mathrm{Syl}_q(G)$. If $C_G(Q) \leq M$, then by Lemmas 2.1 and 2.2, the principal q-block $B_0(G)_q$ of G is the unique q-block of G covering $B_0(M)_q$. In particular, $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_0(G)_q)$. Furthermore we have $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_0(G)_p)$, by ([28], Chap. V, Corollary 5.6). It follows that

$$\{1_G\} \neq \mathrm{Irr}(G/M) \subseteq \mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_q) = \{1_G\},$$

a contradiction. Hence $C_G(Q) \nleq M$, and so $|C_G(Q) : C_M(Q)| = p$. Since $P_0 \leq C_M(Q)$, we conclude that $C_G(Q)$ contains a Sylow p-subgroup of G and we are done. □

Proof of Theorem 1.6

Let G be a minimal counter-example with respect to $|G|$. By Theorem 1.4, G is almost simple, and if we write S for the socle of G then $pq \mid |S|$ and $S \subseteq \mathrm{O}_{p^*,p}(G) \cap \mathrm{O}_{q^*,q}(G)$. By Schreier's conjecture, which has an affirmative answer due to the classification of finite simple groups, we know that G/S is solvable.

(1) $G = \mathrm{O}_{p^*,p}(G) = \mathrm{O}_{q^*,q}(G)$.

Proof. Suppose that $\mathrm{O}_{p^*,p}(G) < G$. Write $K^p = \mathrm{O}_{p^*,p}(G)$ and let M be a maximal normal subgroup of G containing K^p. According to Lemma 3.4 (2) we have $\mathrm{Irr}(B_0(M)_p) \cap \mathrm{Irr}(B_0(M)_q) = \{1\}$. Hence M has a nilpotent Hall $\{p,q\}$-subgroup, say $P_0 \times Q$, where $P_0 \in \mathrm{Syl}_p(M)$ and $Q \in \mathrm{Syl}_q(M)$. By Lemma 3.4 (1), Q is a Sylow q-subgroup of G. If $p \nmid |G/M|$ then P_0 is also a Sylow p-subgroup of G, a contradiction. If $|G/M| = p$, then by Lemma 3.6, G has a nilpotent Hall $\{p,q\}$-subgroup, a contradiction. Thus we have $\mathrm{O}^{p'}(G/K^p) = G/K^p$ and $\mathrm{O}^p(G/K^p) = G/K^p$, which is impossible since G/S is solvable. Thus $G = \mathrm{O}_{p^*,p}(G)$, and by symmetry $G = \mathrm{O}_{q^*,q}(G)$ □

(2) $G = \mathrm{O}_{p^*}(G) = \mathrm{O}_{q^*}(G)$.

Proof. By Step (1) we may choose $S \leq M \trianglelefteq G$ with $\mathrm{O}_{p^*}(G) \leq M$ and $|G/M| = p$. According to Lemma 3.4 (2) we have $\mathrm{Irr}(B_0(M)_p) \cap \mathrm{Irr}(B_0(M)_q) = \{1\}$ and so M has a nilpotent Hall $\{p,q\}$-subgroup. Now Lemma 3.6 implies the existence of a Hall $\{p,q\}$-subgroup of G, a contradiction. □

(3) $G = S\,\mathrm{O}_{p'}(C_G(P)) = S\,\mathrm{O}_{q'}(C_G(Q))$ *where* $P \in \mathrm{Syl}_p(G)$ *and* $Q \in \mathrm{Syl}_q(G)$.

Proof. According to ([20], Chap. X, Theorem 14.17) we have

$$G = \mathrm{O}_{q^*}(G) = \mathrm{O}_{q',E}(G)\,\mathrm{O}_{q^*}(C_G(Q)) = \mathrm{O}_E(G)\,\mathrm{O}_{q^*}(C_G(Q))$$

where $Q \in \mathrm{Syl}_q(\mathrm{O}_{q',E,q}(G))$ and $\mathrm{O}_{q',E}$ and $\mathrm{O}_{q',E,q}$ are as in ([20], Chap. X, Definition 14.17). Since $\mathrm{O}_E(G) = S$ we see that

$$G = \mathrm{O}_{q^*}(G) = S\,\mathrm{O}_{q^*}(C_G(Q)).$$

Applying ([20], Chap. X, Theorem 14.18) and the first sentence of the following Remarks 14.19 we obtain

$$G = \mathrm{O}_{q^*}(G) = S\,\mathrm{O}_{q'}(C_G(Q))$$

where $Q \in \mathrm{Syl}_q(S)$. Applying the same arguments for p instead of q leads to

$$G = \mathrm{O}_{p^*}(G) = S\,\mathrm{O}_{p'}(C_G(P))$$

where $P \in \mathrm{Syl}_p(S)$.

Finally, the assertion $S < G$ follows by ([10], Theorem 1.2). □

4. Almost simple groups

In this section we prove Theorem 1.5, starting with the following observation.

Lemma 4.1. *Let p be a prime, and let N be a normal subgroup of a finite group G such that $|G/N| = r$ is a prime and $G = NC_G(P)$, where $P \in \mathrm{Syl}_p(G)$. Suppose that $p \neq r \neq 2$ and there exists an r-rational G-invariant character $\chi \in \mathrm{Irr}(B_0(N)_p)$. Then the following holds true.*

a) *χ has a unique r-rational extension $\hat{\chi} \in \mathrm{Irr}(G)$.*
b) *$\hat{\chi} \in \mathrm{Irr}(B_0(G)_p)$ and $\mathbb{Q}(\hat{\chi}) = \mathbb{Q}(\chi)$.*

Proof. a) This is ([21], Theorem 6.30).
b) By a) we write $\hat{\chi}$ for the unique r-rational extension of χ to G, and so

$$\chi^G = \sum_{\xi \in \mathrm{Irr}(G/N)} \hat{\chi}\xi.$$

Clearly one of the extensions $\hat{\chi}\xi$ must be in the principal p-block. Suppose that that character is not the r-rational one which we denoted by $\hat{\chi}$. Note that all characters in $\{\hat{\chi}\xi \mid 1 \neq \xi \in \mathrm{Irr}(G/N)\}$ are algebraically conjugate by elements in $\mathrm{Gal}(\mathbb{Q}_{|G|} : \mathbb{Q}_{|G|_{r'}})$. Thus, by Brauer ([7], Lemma 2), $\hat{\chi}\xi \in \mathrm{Irr}(B_0(G)_p)$ for all $1 \neq \xi \in \mathrm{Irr}(G/N)$. Since $r \geq 3$, this contradicts a result of Alperin ([2], Lemma 1). Hence $\hat{\chi} \in \mathrm{Irr}(B_0(G)_p)$. Note that $\mathrm{Gal}(\mathbb{Q}_{|G|} : \mathbb{Q}(\chi))$ stabilizes $\hat{\chi}$, again by ([2], Lemma 1). This implies $\mathbb{Q}(\hat{\chi}) = \mathbb{Q}(\chi)$. □

Remark 4.2. Lemma 4.1 fails if $r = 2$. In this case a rational irreducible character in the principal block of N has two extensions which could both be rational and therefore not distinguishable. But only one of them lies in the principal block.

Proposition 4.3. *Let H be a finite simple group of Lie type over $\mathbb{F}_q$ (here q is a power of a prime) and let $p_1 \neq p_2$ be primes with $p_i \nmid q$. Then $\mathrm{Irr}(B_0(H)_{p_1}) \cap \mathrm{Irr}(B_0(H)_{p_2})$ contains a nontrivial rational unipotent character χ.*

Proof. By ([23], Lemma 3.6), we know that $\mathrm{Irr}(B_0(H)_{p_1}) \cap \mathrm{Irr}(B_0(H)_{p_2})$ contains nontrivial unipotent characters. There remains to prove that at least one such character is rational.

If H is of type

$$A, B, C, D, {}^2A, {}^2D,$$

then all unipotent characters are rational, by ([24], Theorem 0.2 and Remark 1.13).

If H is of type

$$G_2, F_4, {}^2B_2, {}^2G_2, {}^2F_4, {}^3D_4,$$

then we may take the Steinberg character for χ, by ([18], Theorem). Note that the Steinberg character is always rational ([11], Proposition 6.4.4) and unipotent ([11], Corollary 7.6.6).

Thus we are left with groups of type

$$E_6, {}^2E_6, E_7, E_8.$$

We deal explicitly with the case $H = E_6(q)$ in Lemma 4.4 and collect the analogous facts for the other groups in tables so that the reader can carry out the relevant Chevie [16] computations. $\square$

Lemma 4.4. *Let E be the finite simple group of type E_6 over $\mathbb{F}_q$ and let $p_1 \neq p_2$ be primes with $p_i \nmid q$. Then $\mathrm{Irr}(B_0(E)_{p_1}) \cap \mathrm{Irr}(B_0(E)_{p_2})$ contains a nontrivial rational unipotent character χ.*

Proof. Let $\mathbf{E}$ be a simple, simply connected algebraic group of type E_6 and F a Frobenius endomorphism of $\mathbf{E}$ such that $E = \mathbf{E}^F/Z(\mathbf{E}^F)$. Notice that unipotent characters of $\mathbf{E}^F$ have $Z(\mathbf{E}^F)$ in its kernel, hence are characters of E. Thus we may assume that $E = \mathbf{E}^F = E_6(q)$.
a) First note that all 25 unipotent characters of E which lie in the principal series (see [11], page 480) are rational.
This is an immediate consequence of ([15], Proposition 5.6) since all principal series characters are rational.
b) We finish the proof of the lemma by showing that a unipotent character belonging to the principal series lies in $\mathrm{Irr}(B_0(E)_{p_1}) \cap \mathrm{Irr}(B_0(E)_{p_2})$.
Note that

$$|E_6(q)| = q^{36}\Phi_1(q)^6\Phi_2(q)^4\Phi_3(q)^3\Phi_4(q)^2\Phi_5(q)\Phi_6(q)^2\Phi_8(q)\Phi_9(q)\Phi_{12}(q)$$

where $\Phi_n(x)$ denotes the n-th cyclotomic polynomial in $\mathbb{Z}[x]$. Thus we may write

$$|E_6(q)| = q^{36}\prod_{i\in\mathcal{I}}\Phi_i^{a(i)}$$

with $\mathcal{I} = \{1, 2, 3, 4, 5, 6, 8, 9, 12\}$. Let

$$e_i = \text{multiplicative order of } q \text{ modulo} \begin{cases} p_i & \text{if } p_i \neq 2 \\ 4 & \text{if } p_i = 2. \end{cases}$$

Clearly, $e_1, e_2 \in \mathcal{I}$. For $i = 1, 2$ let $\mathbf{S}_i$ be a Sylow e_i-torus of $\mathbf{E}$ and let $\mathbf{L}_i = \mathbf{C}_{\mathbf{E}}(\mathbf{S}_i)$. Then $(\mathbf{L}_i, 1_{\mathbf{L}_i^F})$ is a unipotent e_i-cuspidal pair and by [13] all irreducible constituents of $R^{\mathbf{E}}_{\mathbf{L}_i}(1_{\mathbf{L}_i^F})$ lie in a fixed p_i-block which must be the principal p_i-block since the trivial character is a constituent of $R^{\mathbf{E}}_{\mathbf{L}_i}(1_{\mathbf{L}_i^F})$. With the help of Chevie [16] we see that we may take the Steinberg character for χ if $e_1 \neq 5 \neq e_2$. Thus, w.l.o.g. we may suppose that $e_1 = 5$.

Now we use Chevie [16] again to compute explicitely the irreducible components of $R^{\mathbf{E}}_{\mathbf{L}_i}(1_{\mathbf{L}_i^F})$. It turns out that for $e_2 = 8$ the unipotent character $\phi_{81,10}$ lies in $R^{\mathbf{E}}_{\mathbf{L}_i}(1_{\mathbf{L}_i^F})$ for $i = 1, 2$, and $\phi_{64,13}$ in case $e_2 = 9$. In all other cases the unipotent character $\phi_{6,25}$ does the job. Note that all these characters belong to the principal series, also the Steinberg character (see [11], Section 13.9). Thus the proof of the lemma is complete. $\square$

In Tables 1,2,3 we deal with the remaining groups $E_7(q)$, $E_8(q)$ and ${}^2E_6(q)$. For particular values e_1, e_2 we determine a unipotent character. If the pair e_1, e_2 is not in the list the Steinberg character may be chosen. Characters of the form $\phi_{a,b}$ are in the principal series, hence rational by Geck's result ([15], Proposition 5.6). In table 3 the chararacter ${}^2A_5, \epsilon$ arises from a cuspidal unipotent character of the Levi subgroup ${}^2A_5(q)$ which is rational by Lusztig's result ([24], Remark 1.13). Thus ${}^2A_5, \epsilon$ is rational, again by Geck's result ([15], Proposition 5.6). In Table 1 and Table 2 there are characters of the form D_4, a for several a. These characters arise from a cuspidal unipotent character of the Levi subgroup $D_4(q)$ which are rational, again by Lusztig's result ([24], Theorem 0.2). The rationality of D_4, a follows by applying Geck's result again.

Proof of Theorem 1.5. Without loss of generality, we may assume that $q = 2$ and p is an odd prime. Let G be a minimal counter-example to Conjecture 1.3. By Theorem 1.6, we have $S < G = S\,\mathrm{O}_{p'}(C_G(P)) = S\,\mathrm{O}_{2'}(C_G(Q))$ for P a Sylow p-subgroup and Q a Sylow 2-subgroup of S. Thus, by ([23], Lemma 2.2), S must be a simple group of Lie type in odd characteristic. Furthermore, if p is the defining characteristic of S, then by ([6], Lemma 2.2) $C_G(P)$ is a p-group, and so $G = S$, a contradiction. Hence neither p nor 2 can be the defining characteristic of S.

Let $S = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_s = G$ where $|N_i/N_{i-1}|$ is an odd prime different from p. By Proposition 4.3, we see that $\mathrm{Irr}(B_0(S)_p) \cap \mathrm{Irr}(B_0(S)_2)$ contains a nontrivial rational unipotent character χ_0. Furthermore note that by Dedekind's identity we have $N_i = N_{i-1}C_{N_i}(P) = N_{i-1}C_{N_i}(Q)$. Now Alperin's result ([2], Lemma 1) says that the restriction map

$$\mathrm{Irr}(B_0(G)_p) \ni \hat{\chi} \longrightarrow \hat{\chi}|_S = \chi \in \mathrm{Irr}(B_0(S)_p)$$

is a bijection. By symmetry the same also holds true for the prime 2. This implies that the character χ_0 extends to G and so to N_1. Therefore, by Lemma 4.1, the unique $|N_1/N_0|$-rational extension ξ of χ_0 to N_1, which is indeed rational since $\mathbb{Q}(\xi) = \mathbb{Q}(\chi_0)$, is contained in $\mathrm{Irr}(B_0(N_1)_p) \cap \mathrm{Irr}(B_0(N_1)_2)$. In particular, the extension ξ is N_2-invariant.

Repeating this step via the chain $S = N_0 \trianglelefteq N_1 \trianglelefteq \cdots \trianglelefteq N_s = G$ we finally obtain $|\mathrm{Irr}(B_0(G)_p) \cap \mathrm{Irr}(B_0(G)_2)| \geq 2$, a contradiction. This completes the proof. $\square$

5. Cyclic Sylow subgroups

Using completely different methods than above we prove Theorem 1.8. Based on Osima's character relation for blocks Feit has shown the following ([14], Chap. IV, Lemma 6.4).

Lemma 5.1. *Let $B(G)_p$ and $B(G)_q$ be any blocks for primes $p \neq q$. If $x \in G$ is a p-element and $y \in G$ is a q-element such that no conjugate of x commutes with any conjugate of y, then*

$$\sum_{\chi \in \mathrm{Irr}(B(G)_p) \cap \mathrm{Irr}(B(G)_q)} \chi(x)\chi(y) = 0.$$

TABLE 1. $E_7(q)$, $\mathcal{I} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 18\}$

e_1	e_2	rational unipotent character
4	1,2,3,4,6,12	$\phi_{210,6}$
	5	$\phi_{336,11}$
	7	$\phi_{405,8}$
	8	$\phi_{105,6}$
	9	$\phi_{56,3}$
	10	D_4, σ_2
	14	$\phi_{189,10}$
	18	$D_4, 1$
5	1,2,3,5,6,18	$\phi_{21,33}$
	7	$\phi_{216,16}$
	8,10	$\phi_{189,22}$
	12	$\phi_{336,1}$
	9	$\phi_{56,30}$
	14	$\phi_{189,17}$
8	1,2,3,6,8,12	$\phi_{21,36}$
	7	$\phi_{216,9}$
	9	$\phi_{280,17}$
	10	$\phi_{189,22}$
	14	$D_4, r\varepsilon_1$
	18	$D_4, r\varepsilon$
10	1,2,3,6,9,10	$\phi_{35,31}$
	7	$\phi_{405,15}$
	12	D_4, σ_2
	14	$D_4, r\varepsilon_2$
	18	D_4, ε
12	1,2,3,6,12	$\phi_{21,36}$
	7	$\phi_{120,25}$
	9	$\phi_{56,3}$
	14	D_4, ε_2
	18	$D_4, 1$

Now suppose that the defect group of $B(G)_p$ is cyclic, hence generated by some $x \in G$. If $\chi \in \mathrm{Irr}(B(G)_p)$, then it may happen that $\chi(x) = 0$, and the above Lemma tells us nothing. However, by an argument we learned from Gabriel Navarro, we get the following.

Lemma 5.2. *Let $\chi \in \mathrm{Irr}(G)$ with $p \nmid \chi(1)$. If $g \in G$ is a p-element, then $\chi(g) \neq 0$.*

Proof. Let p^n be the order of g. Note that g is a zero of the polynomial $x^{p^n} - 1$ which has only simple roots. Thus a suitable matrix of g with respect to χ is a diagonal matrix where the elements in the diagonal are p^n-th roots of unity. Let M be a maximal ideal of

TABLE 2. $E_8(q)$, $\mathcal{I} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 24, 30\}$

e_1	e_2	rational unipotent character
7	1,2,3,5,6,7,10,14,15,30	$\phi_{8,91}$
	4,12	$\phi_{400,7}$
	8,24	$\phi_{160,55}$
	9	$\phi_{3200,22}$
	18	$\phi_{2400,23}$
	20	$\phi_{1296,33}$
9	1,2,3,5,6,9,10,30	$\phi_{28,68}$
	4,12,15,20	$\phi_{112,63}$
	8,24	$\phi_{160,7}$
	14	$\phi_{5600,21}$
	18	$\phi_{1008,9}$
14	1,2,3,5,6,10,14,15,30	$\phi_{8,91}$
	4,8	$\phi_{6075,14}$
	12	$\phi_{210,52}$
	18	$D_4, \phi'_{8,9}$
	24	$D_4, \phi''_{2,16}$
18	1,2,3,5,6,10,15,18	$\phi_{84,64}$
	4,12	$\phi_{210,4}$
	8,24	$D_4, \phi'_{2,4}$
	20,30	$D_4, \phi_{1,24}$

TABLE 3. ${}^2E_6(q^2)$, $\mathcal{I} = \{1, 2, 3, 4, 6, 10, 12, 18\}$

e_1	e_2	rational unipotent character
10	1,2,3,4,6,10,12	$\phi''_{2,16}$
	18	${}^2A_5, \varepsilon$

the ring R of algebraic integers containing pR. Clearly, if we reduce a p^n-th root of unity mod M we get 1. This shows that

$$\chi(g) \equiv \chi(1) \bmod M.$$

Since $p \nmid \chi(1)$ we finally obtain $\chi(g) \neq 0$. □

Proof of Theorem 1.8. Let $B(G)_p$ and $B(G)_q$ be blocks of maximal defect. Let $P = \langle x \rangle$ be a Sylow p-subgroup of G and let $Q = \langle y \rangle$ be a Sylow q-subgroup of G. Suppose that $\chi \in \mathrm{Irr}(B(G)_p) \cap \mathrm{Irr}(B(G)_q)$. Since all irreducible characters in a block with a cyclic defect group are of height 0, we get $p \nmid \chi(1)$ and $q \nmid \chi(1)$. Thus, by Lemma 5.2, we see that $\chi(x)\chi(y) \neq 0$. If $[x^g, y^h] \neq 1$ for all $g, h \in G$, then we may apply Lemma 5.1 and see that $|\mathrm{Irr}(B(G)_p) \cap \mathrm{Irr}(B(G)_q)| \geq 2$, a contradiction. Thus there exists $h \in G$ such that $[x, y^h] = 1$, which proves the theorem. □

6. Symmetric groups

In this section we prove Theorem 1.9. For convenience, we start by recalling some facts about partitions from [29]. Let t, n be positive integers and let λ, μ be partitions of n. A *t-core partition* of n is a partition of n all of whose hook numbers are not divisible by t, and the t-core $\lambda_{(t)}$ of λ is the t-core partition obtained by recursively removing all t-hooks starting from λ.

A *partition sequence* is a bi-infinite sequence of zeroes and ones, such that if we consider the sequence going from the left to the right we have:

(i) All entries to the left of a certain point are zeroes.
(ii) All entries to the right of a certain point are ones.

For example,

$$\ldots\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ \ldots$$

is a partition sequence, where the dots on the left and the right represent infinite sequences of zeroes and ones, respectively. By [29, Lemma 2.1], any partition can be uniquely determined by a partition sequence.

Next, we recall some basic facts about partition sequences, which can be found in [1, Section 3] and [30, Exercise 7.59]. Visually, the partition sequence of a partition λ can be obtained as follows: For the edges in the boundary of the Young diagram of λ, starting at the bottom and ending to the right, we label the vertical (resp. horizontal) edges with 0 (resp. 1) (see Figure 1). In this way, we get a 01-sequence Λ_λ. Adding an infinite sequence of zeroes to the left of Λ_λ and an infinite sequence of ones to the right of Λ_λ, we obtain a partition sequence, which is indeed the one associated with the partition λ in [29, Lemma 2.1].

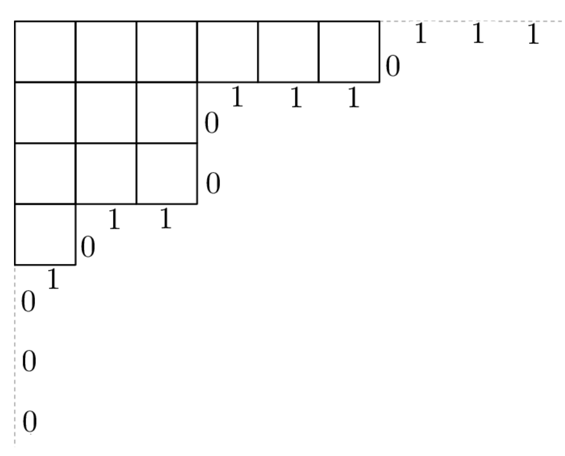

Figure 1. The partition sequence of a partition $\lambda = (6, 3, 3, 1)$

Given a partition λ and its 01-sequence Λ_λ, we may find a certain entry, which will be denoted by z_0, in Λ_λ such that the number of ones to the left side of z_0 (not including z_0) is equal to the number of zeroes to the right side of z_0 (including z_0 itself if and only if $z_0 = 0$). Now we may number the entries of the partition sequence Λ of λ increasing to the right side of z_0 and decreasing to the left side of z_0, so that $\Lambda = (z_i)_{i\in\mathbb{Z}}$.

Note that

$$|\lambda| = |\{(i,j) : i < j,\ z_i = 1,\ z_j = 0\}|.$$

Actually, each box in λ is uniquely determined by such a pair (i,j) (which differs from the coordinate of the box). The hook length of such a box is $j - i$. Therefore, the partition λ is a t-core partition if and only if there is no $i \in \mathbb{Z}$ satisfying $z_i = 1$, $z_{i+t} = 0$ in its partition sequence $\Lambda = (z_i)_{i\in\mathbb{Z}}$. In addition, if $z_i = 1$, $z_{i+t} = 0$, by exchanging the values of z_i and z_{i+t} we get a new partition sequence and thus a new partition (which is obtained by removing a *t-hook* from λ). Correspondingly, if $z_i = 0$, $z_{i+t} = 1$, by exchanging the values of z_i and z_{i+t} we get a new partition (which is obtained by adding a t-hook to λ). The following observation is now obvious.

Lemma 6.1. *For any partition λ and a positive integer t, let $w_t(\lambda) = (|\lambda| - |\lambda_{(t)}|)/t$. Then*

$$w_t(\lambda) = |\{(i,k) : z_i = 1, z_{i+kt} = 0, i \in \mathbb{Z}, k \in \mathbb{N}\}|.$$

Also, the following lemma is well-known, and we will use it freely.

Lemma 6.2. *Let λ be a partition with partition sequence $\Lambda = (z_i)_{i\in\mathbb{Z}}$. If λ is not a t-core partition for some positive integer t, then λ has a hook length equal to t, or alternatively, there exists some integer e such that $(z_e, z_{e+t}) = (1, 0)$.*

Proof. Since λ is not a t-core partition, there exists $(z_i, z_{i+kt}) = (1, 0)$ for some $i \in \mathbb{Z}$ and $k \in \mathbb{N}$. Suppose $k \geq 2$. If $z_{i+(k-1)t} = 1$, then we let $e = i+(k-1)t$ so that $(z_e, z_{e+t}) = (1, 0)$. In the case that $z_{i+(k-1)t} = 0$ we consider the pair $(z_i, z_{i+(k-1)t}) = (1, 0)$ and repeat the process to finally get the claimed e. $\square$

Now let p and q be different primes. Let λ be a partition of n. We say that λ is a *(p,q)-isolated partition* if there does not exist any other partition of n with the same p- and q-core as λ.

Lemma 6.3. *Let λ be a partition with partition sequence $(z_i)_{i\in\mathbb{Z}}$. If λ is (p,q)-isolated for different primes p and q, then there are no $a, b, c, d \in \mathbb{Z}$ such that $(z_a, z_b) = (1, 0)$, $(z_c, z_d) = (0, 1)$, $a \equiv b \pmod p$, $a \equiv c \pmod q$, and $b - a = d - c > 0$.*

Proof. Assume that there are such $a, b, c, d \in \mathbb{Z}$ as in the lemma. We will show that λ is not a (p,q)-isolated partition, which leads to a contradiction.

Observe that $d \equiv b \pmod q$ since $b - a = d - c$ and $a \equiv c \pmod q$. If $a = d$ or $b = c$, we have $a \equiv d \equiv b \equiv c \pmod q$. However, since $a \equiv b \pmod p$, it follows that

$$a \equiv b \pmod{pq}.$$

The condition $(z_a, z_b) = (1, 0)$ implies that λ has a hook length pq, hence it is not a pq-core. Removing the hook and then adding a hook of length pq either vertically to the left of the partition or horizontally to the right of it, we see that there exists some other partition μ of $|\lambda|$ with the same pq-core as that of λ. Therefore

$$\lambda_{(p)} = (\lambda_{(pq)})_{(p)} = (\mu_{(pq)})_{(p)} = \mu_{(p)},$$

Similarly, we have $\lambda_{(q)} = \mu_{(q)}$, hence λ is not (p, q)-isolated.

Thus we may assume $a \neq d$ and $b \neq c$. Let μ be the partition obtained by interchanging (z_a, z_b) and (z_c, z_d), so that $\mu \neq \lambda$. Since $b - a = d - c$, we have $|\mu| = |\lambda|$ and μ has the same p- and q-core as λ. Thus λ is again not (p, q)-isolated, as wanted. □

We say that a partition sequence is (p, q)-*incongruent* if it satisfies the conclusion of Lemma 6.3. So Lemma 6.3 asserts that the partition sequence of a (p, q)-isolated partition is (p, q)-incongruent.

Lemma 6.4. *Let λ be a partition with partition sequence $\Lambda = (z_i)_{i\in\mathbb{Z}}$. If Λ is (p, q)-incongruent, then $|\lambda| \leq |\lambda_{(p)}| + |\lambda_{(q)}|$.*

Proof. We argue by induction on $w_q(\lambda) = (|\lambda| - |\lambda_{(q)}|)/q$. Clearly, Lemma 6.4 holds for $w_q(\lambda) = 0$ since $\lambda = \lambda_{(q)}$ in this case.

Now we assume $w_q(\lambda) \geq 1$ so that λ is not a q-core partition. In particular, there exists some integer e such that $(z_e, z_{e+q}) = (1, 0)$. Since Λ is (p, q)-incongruent, there does not exist any integer e' such that $e' \equiv e \pmod p$ and $(z_{e'}, z_{e'+q}) = (0, 1)$. Hence for any integer f with $f \equiv e \pmod p$, the pair (z_f, z_{f+q}) has at most three possibilities: $(1, 0)$, $(0, 0)$ or $(1, 1)$. Interchanging the values of all pairs (z_f, z_{f+q}) with $(z_f, z_{f+q}) = (1, 0)$ and $f \equiv e \pmod p$ in Λ, or equivalently, (index-disjoint) subsequences $\{z_{e+kp} : k \in \mathbb{Z}\}$ and $\{z_{e+q+kp} : k \in \mathbb{Z}\}$ in Λ, we obtain a new partition sequence, say $\Lambda' = (z'_i)_{i\in\mathbb{Z}}$, which satisfies

$$z'_i = \begin{cases} z_{i+q}, & \text{if } i \equiv e \pmod p \\ z_{i-q}, & \text{if } i \equiv e + q \pmod p \\ z_i, & \text{otherwise.} \end{cases}$$

Let λ' be the partition corresponding to Λ'. By Lemma 6.1, we have $w_p(\lambda') = w_p(\lambda)$ but $w_q(\lambda') < w_q(\lambda)$.

We claim that Λ' is (p, q)-incongruent. Otherwise, there exist some $a, b, c, d \in \mathbb{Z}$ such that $(z'_a, z'_b) = (1, 0)$, $(z'_c, z'_d) = (0, 1)$, $a \equiv b \pmod p$, $a \equiv c \pmod q$, and $b - a = d - c$.

If $a \equiv e \pmod p$ and $c \not\equiv e$ or $e + q \pmod q$, then

$$(z_{a+q}, z_{b+q}) = (z'_a, z'_b) = (1, 0) \text{ and } (z_c, z_d) = (z'_c, z'_d) = (0, 1).$$

Clearly

$$a + q \equiv b + q \pmod p, \ \ a + q \equiv c \pmod q \ \text{ and } \ (b + q) - (a + q) = d - c.$$

It follows that Λ is not (p, q)-incongruent, a contradiction. For the other cases, we similarly get a contradiction, whence the claim follows.

Thus, by the inductive hypothesis on λ', we have

$$|\lambda'| \leq |\lambda'_{(p)}| + |\lambda'_{(q)}|.$$

Furthermore, since λ' can be obtained from λ by removing some q-hooks , we have

$$\lambda'_{(q)} = \lambda_{(q)}.$$

In addition, it follows from $w_p(\lambda') = w_p(\lambda)$ that

$$|\lambda'| - |\lambda'_{(p)}| = |\lambda| - |\lambda_{(p)}|.$$

Thus

$$|\lambda| \leq |\lambda_{(p)}| + |\lambda_{(q)}|.$$

□

Proof of Theorem 1.9. We first recall some facts about the representation theory of symmetric groups. It is well known that the complex irreducible characters of the symmetric group S_n of degree n are naturally labeled by the partitions of n. Thus we may denote by χ_λ the complex irreducible characters of G labeled by λ. Now let p be a prime. For partitions λ and μ of n, the already proved Nakayama Conjecture [29, Theorem 11.1] asserts that χ_λ and χ_μ lie in the same p-block of S_n if and only if they have the same p-core. So the p-blocks of S_n are parameterized by the p-cores of partitions of n. Let B be a p-block of S_n corresponding to a p-core κ, and write $w = (n - |\kappa|)/p$. According to [29, Proposition 11.3], the p-block B has a defect group which is conjugate in S_n to a Sylow p-subgroup of $\mathrm{S}_{pw} \leq \mathrm{S}_n$. (For further details, see [22] or [29].)

Now let q be a prime different from p. Suppose that for some partition λ of n, the corresponding character $\chi_\lambda \in \mathrm{Irr}(\mathrm{S}_n)$ satisfies $\mathrm{Irr}(B(\mathrm{S}_n)_p) \cap \mathrm{Irr}(B(\mathrm{S}_n)_q) = \{\chi_\lambda\}$. This is equivalent to saying that the partition λ is (p,q)-isolated. Therefore, by Lemmas 6.3 and 6.4, we have

$$n \leq |\lambda_{(p)}| + |\lambda_{(q)}|,$$

or equivalently

$$(n - |\lambda_{(p)}|) + (n - |\lambda_{(q)}|) \leq n.$$

It follows that S_n has a subgroup isomorphic to $\mathrm{S}_{pw_1} \times \mathrm{S}_{qw_2}$, where $pw_1 = n - |\lambda_{(p)}|$ and $qw_2 = n - |\lambda_{(q)}|$. In particular, the Sylow p-subgroups of S_{pw_1} commute with the Sylow q-subgroups of S_{qw_2}. However, they are defect groups of $B(\mathrm{S}_n)_p$ and $B(\mathrm{S}_n)_q$, respectively. Thus the theorem is proved. □

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