ON MORITA THEORY FOR SELF-DUAL MODULES

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Abstract. Let G be a finite group and let k be a field of characteristic p. It is known that a kG-module V carries a non-degenerate G-invariant bilinear form b if and only if V is self-dual. We show that whenever a Morita bimodule M which induces an equivalence between two blocks $B(kG)$ and $B(kH)$ of group algebras kG and kH is self-dual then the correspondence preserves self-duality. Even more, if the bilinear form on M is symmetric then for p odd the correspondence preserves the geometric type of simple modules. In characteristic 2 this holds also true for projective modules.

1. Introduction

Let k be a field and let G be a finite group. It is well-known that a kG-module V, simply called a G-module, carries a non-degenerate G-invariant k-bilinear form if and only if V is isomorphic to its dual $V^* = Hom_k(V, k)$. Up to an automorphism of V, a simple self-dual G-module $V \cong V^*$ carries exactly one G-invariant bilinear form different from the zero. In case k is large enough, for instance k algebraically closed, this means that such a form, if it exists, is unique up to a scalar. If in addition the characteristic p of k is odd this form is symmetric or antisymmetric. The case $p = 2$ turns out to be more subtle. Note that in this case an antisymmetric form is symmetric. By Fong's Lemma ([4], Chap. VII, Theorem 8.13), a simple self-dual G-module V different from the trivial module always carries a non-degenerate G-invariant symplectic form. However, it may happen that V even has a non-degenerate G-invariant quadratic form. This holds true, for instance, if G is solvable but not in general. What really happens is not an easy task to decide; it is a question in cohomology (see [11]), at least for arbitrary groups. To be brief throughout the paper we call a G-module of symmetric, antisymmetric, symplectic, resp. quadratic type if V carries a non-degenerate G-invariant form of symmetric, antisymmetric, symplectic, resp. quadratic type. The reader not familiar with duality theory of modules may refer Chapter VII of [4].

In representation theory of finite groups G we are often interested in the category of modules belonging to a p-block B_G of kG where the underlying field k is of characteristic p. Instead of investigating B_G we may study a Morita equivalent p-block B_H for some other group H. Recall that one calls two blocks Morita equivalent if their module categories are equivalent as exact categories. Morita theory shows that the module categories of the two blocks are equivalent if and only if there is an $H \times G$-bimodule $M = {}_HM_G$ where M is a progenerator for B_H and for B_G. Moreover the functor

$$V \mapsto M \otimes_{kG} V$$

defines an equivalence from the module category of B_G onto that of B_H. The correspondence preserves all functorial properties of module categories, in particular being simple, indecomposable, projective, Loewy structure and the like. For more details the reader is referred to Chapter 4 in [10].

Date: May 12, 2007.

The second author is supported by a German-French grant "partenariat Hubert Curien PROCOPE dossier 14765WB" and gratefully acknowledges the financial support.

In this note we study the behaviour of self-duality including the type of forms under Morita equivalence. We start with a series of examples which illuminates the situation.

2. Examples

In this section we show that Morita equivalence does not need to preserve self-duality of simple modules. In case it does, examples show that the geometric type of simple modules may vary via the correspondence.

In order to state the examples we need that blocks are determined by their Brauer trees up to Morita equivalence. Kupisch determined the structure of indecomposable modules of a Brauer tree algebra, and hence he implicitly determined their module category, i.e., the Morita equivalence class of a Brauer tree algebra. In modern terms Kupisch's result reads as follows.

Theorem 2.1. (Kupisch [7, 8]) *Let A_1 and A_2 be two Brauer tree algebras. Then A_1 and A_2 are Morita equivalent if and only if their Brauer trees, including the exceptional vertex and its multiplicity, coincide.*

In the following examples we shall use Brauer trees computed by Hiss and Lux in [5]. The geometric type of simple self-dual modules, i.e. symmetric or alternating, is taken from [1] or [6].

Example 2.2. Let k be an algebraically closed field of characteristic 3 and let $G = SL_2(5)$ be the double cover of the alternating group $\mathfrak{A}_5$ of degree 5. Then a non principal 3-block, say $B_1(G)$, of G is a Brauer tree algebra corresponding to the Brauer tree

$$T: \quad \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet$$

without an exceptional vertex. The ordinary characters of the algebra have dimensions 2, 4 and 2. Both of the simple modules in characteristic 3 of $B_1(G)$ admit an, up to scalar unique, G-invariant bilinear form. This form is alternating for both modules.

The principal 3-block of the alternating group $H = \mathfrak{A}_5$ is also a Brauer tree algebra with Brauer tree T. The ordinary characters of this algebra have dimensions 1, 5 and 4. In characteristic 3 both simple modules admit an, up to scalar unique, $\mathfrak{A}_5$-invariant bilinear form which is symmetric in both cases.

As a consequence, the type of bilinear forms is usually not preserved by Morita equivalence, even if self-duality is preserved on simple modules.

Example 2.3. Let k be again an algebraically closed field but of characteristic 7 and let $G = 3McL$ be the triple cover of a the McLaughlin simple group McL.

Then, a non principal 7-block, say $B_2(3McL)$, of G is a Brauer tree algebra corresponding to the Brauer tree

$$S: \quad \bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ_2$$

with an exceptional vertex $\circ_2$ of multiplicity 2. The ordinary characters of the algebra have dimensions 792, 4752, 6336 and 2376 read from the left in the order of the tree. None of the three simple modules in characteristic 7 is self-dual, hence none of them admit a G-invariant bilinear form.

The principal 7-block $B_3(M_{23})$ of the Mathieu group $H = M_{23}$ is a Brauer tree algebra also with Brauer tree S. The ordinary characters of the algebra have dimensions 1, 1035, 2024 and 990 read from the left. The exceptional vertex corresponds to the two ordinary characters of degree 990. All simple modules in characteristic 7 admit an, up to scalar unique, H-invariant bilinear form which is symmetric in all three cases.

As a consequence every Morita bimodule inducing a Morita equivalence between $B_3(M_{23})$ and $B_2(3McL)$ has the property that each self-dual simple $B_3(M_{23})$-module is sent to a simple $B_2(3McL)$-module which is not self-dual.

3. Self-dual Morita bimodules

Let H and G be finite groups and let k be a field. Furthermore, let M be an $H \times G$-bimodule where H acts on the left and G on the right. Then $M^* = Hom_k(M,k)$ becomes an $H \times G$-bimodule by setting

$$(hfg)m = f(h^{-1}mg^{-1})$$

for $h \in H, g \in G, f \in M^*$ and $m \in M$. Indeed, the module structure follows by

$$\begin{aligned} ((h_2h_1)f(g_1g_2))m &= f((h_2h_1)^{-1}m(g_1g_2)^{-1}) \\ &= f(h_1^{-1}h_2^{-1}mg_2^{-1}g_1^{-1}) \\ &= (h_1fg_1)(h_2^{-1}mg_2^{-1}) \\ &= (h_2(h_1fg_1)g_2)m. \end{aligned}$$

Definition 3.1. *Let M be an $H \times G$-bimodule. A k-bilinear form $(\cdot,\cdot)$ on M is called $H \times G$-invariant if*

$$(hmg, hng) = (m,n)$$

for all $h \in H$, all $g \in G$ and all $m,n \in M$.

The next lemma is well-known. For the readers convenience we give the proof.

Lemma 3.2. *Let M be an $H \times G$-bimodule. Then M is isomorphic to M^* as an $H \times G$-bimodule if and only if M carries a non-degenerate $H \times G$-invariant bilinear form. More precisely, if $\varphi : M \to M^*$ is an isomorphism, then $(m,m') := \varphi(m)(m')$ is a non-degenerate $H \times G$-invariant bilinear form. If $(\cdot,\cdot)$ is a non-degenerate $H \times G$-invariant bilinear form, then the map $M \ni m \mapsto (\cdot,m) \in M^*$ is an isomorphism of $H \times G$-bimodules.*

Proof. If $\varphi : M \to M^*$ denotes an $H \times G$-isomorphism we put

$$(m,n) := \varphi(m)n$$

for $m,n \in M$. Clearly, the form $(\cdot,\cdot)$ is k-linear, but it is also $H \times G$-invariant since

$$\begin{aligned} (hmg, hng) &= \varphi(hmg)(hng) \\ &= (h\varphi(m)g)(hng) && \text{(since } \varphi \text{ is } H \times G\text{-linear)} \\ &= \varphi(m)(h^{-1}(hng)g^{-1}) && \text{(the module structure of } M^*) \\ &= \varphi(m)n \\ &= (m,n). \end{aligned}$$

In addition $(\cdot,\cdot)$ is non-degenerate since φ is an isomorphism.

Conversely, if $(\cdot,\cdot)$ is a non-degenerate $H \times G$-invariant bilinear form on M then we put

$$\varphi(m)n := (m,n)$$

for all $m,n \in M$. Note that φ is a k-isomorphism since the bilinear form is non-degenerate. Furthermore, φ is $H \times G$-linear since

$$(hmg, n) = \varphi(hmg)n$$

and

$$(hmg,n) = (m, h^{-1}ng^{-1}) = \varphi(m)(h^{-1}ng^{-1}) = (h\varphi(m)g)n$$

for all $h \in H, g \in G$ and $m,n \in M$. ∎

Proposition 3.3. *Let M be a self-dual $H \times G$-bimodule which is projective as a G-module. If V is a self-dual G-left module then $M \otimes_{kG} V$ is a self-dual H-left module. More precisely, if $\beta : M \to M^*$ is an $H \times G$-linear isomorphism and $\alpha : V \to V^*$ is a G-linear isomorphism, then*

$$\begin{array}{ccc} M \otimes_{kG} V & \longrightarrow & (M \otimes_{kG} V)^* \\ m \otimes v & \mapsto & \left(m' \otimes v' \mapsto \sum_{g \in G} \beta(mg^{-1})(m') \cdot \alpha(v)(g^{-1}v') \right) \end{array}$$

is an H-linear isomorphism.

Proof. Let $\beta : M \to M^*$ denote an $H \times G$-isomorphism and $\alpha : V \to V^*$ a G-isomorphism. Therefore, since α and β are kG-linear the map $\gamma := \beta \otimes_{kG} \alpha : M \otimes_{kG} V \to M^* \otimes_{kG} V^*$ is a well-defined k-vector space isomorphism satisfying $(\beta \otimes_{kG} \alpha)(m \otimes_{kG} v) = \beta(m) \otimes_{kG} \alpha(v)$ for all $m \in M$ and $v \in V$ (cf. e.g. [2, Theorem 12.10]).

Now, let $m \in M, v \in V$ and $h \in H$. Since

$$\begin{aligned} \gamma(h(m \otimes_{kG} v)) &= \gamma(hm \otimes_{kG} v) \\ &= \beta(hm) \otimes_{kG} \alpha(v) \\ &= h\beta(m) \otimes_{kG} \alpha(v) \quad \text{(since } \beta \text{ is } H\text{-linear)} \\ &= h\gamma(m \otimes_{kG} v). \end{aligned}$$

the map γ is H-linear.

Finally, we claim that

$$M^* \otimes_{kG} V^* \cong (M \otimes_{kG} V)^*$$

as H-left modules. We prove this via a sequence of isomorphisms.

Observe that V^* is not only a G-left but in a natural way also a G-right module via the definition

$$(fg)v = f(gv)$$

for $f \in V^*, g \in G$ and $v \in V$.

Considering V^* as G-right module $Hom_{kG}(M, V^*)$ makes sense. With this observation we have

Claim (i) $Hom_{kG}(M, V^*) \cong Hom_k(M \otimes_{kG} V, k) = (M \otimes_{kG} V)^*$ as H-left modules.

The isomorphism is given by sending $\alpha \in Hom_{kG}(M, V^*)$ to the map $\Phi(\alpha) : M \otimes_{kG} V \to k$ with $\Phi(\alpha)(m \otimes_{kG} v) = \alpha(m)v$ where $m \in M$ and $v \in V$. Note, that Φ is well-defined since α is G-linear, for

$$\Phi(\alpha)(mg \otimes_{kG} v) = \alpha(mg)(v) = (\alpha(m)g)\,(v) = \alpha(m)(gv) = \Phi(\alpha)(m \otimes_{kG} gv).$$

Furthermore, Φ is H-linear since

$$\Phi(h\alpha)(m \otimes_{kG} v) = (h\alpha)(m)v = \alpha(h^{-1}m)v = \Phi(\alpha)(h^{-1}m \otimes_{kG} v) = (h\Phi(\alpha))(m \otimes_{kG} v).$$

Finally, $\alpha \to \Phi(\alpha)$ is obviously a monomorphism. In order to show that Φ is an isomorphism we define a k-linear map

$$\Psi : Hom_k(M \otimes_{kG} V, k) \to Hom_{kG}(M, V^*)$$

by putting $(\Psi(\varphi)(m))(v) := \varphi(m \otimes v)$ for $\varphi \in Hom_k(M \otimes_{kG} V, k)$, $m \in M$ and $v \in V$. Since φ is defined over $M \otimes_{kG} V$ we have

$$(\Psi(\varphi)(mg))v = \varphi(mg \otimes_{kG} v) = \varphi(m \otimes_{kG} gv) = (\Psi(\varphi)(m))(gv) = (\Psi(\varphi)(m)g)v$$

for $m \in M, g \in G$ and $v \in V$. Thus $\Psi(\varphi) \in Hom_{kG}(M, V^*)$. Moreover,

$$((\Phi \circ \Psi)(\varphi))(m \otimes -) = \Phi(m \mapsto \varphi(m \otimes -)) = \varphi(m \otimes -)$$

for $m \in M$. Thus Φ is surjective, hence an isomorphism.

Claim (ii) $Hom_{kG}(M, kG) \otimes_{kG} V^* \cong Hom_{kG}(M, V^*)$ as H-left modules:

The isomorphism is given by

$$\alpha \otimes_{kG} f \mapsto \widehat{\alpha \otimes_{kG} f}$$

for $\alpha \in Hom_{kG}(M, kG)$ and $f \in V^*$ where

$$\left((\widehat{\alpha \otimes_{kG} f})(m)\right)(v) = (f\alpha(m))(v) = f\left(\alpha(m)v\right)$$

for $m \in M$ and $v \in V$. Here V^* is considered as a G-right module by the natural action as defined above. Furthermore, the space $Hom_{kG}(M, kG)$ is an $H \times G$-bimodule via the action

$$(hfg)(m) := g^{-1}f(h^{-1}m)$$

for $m \in M$, $g \in G$, $h \in H$ and $f \in Hom_{kG}(M, kG)$.

First we show that the map $\widehat{\quad}$ is well-defined.

Indeed, for $f \in V^*$, $g \in G$, $\alpha \in Hom_{kG}(M, kG)$, $v \in V$ and $m \in M$ one have

$$\begin{aligned}
\left((\widehat{(\alpha g) \otimes_{kG} f})(m)\right)(v) &= \left(f((\alpha g)(m))\right)(v) && \text{(definition of } \widehat{\quad}\text{)}\\
&= f\left(((\alpha g)(m))\, v\right) && \text{(right module structure of } V^*\text{)}\\
&= f\left(\left(g^{-1}(\alpha(m))\right) v\right) && \text{(right module structure of } Hom_{kG}(M, kG)\text{)}\\
&= f\left(g^{-1}\left(\alpha(m)v\right)\right) && \text{(associativity of the } kG\text{-action on } V^*\text{)}\\
&= (gf)(\alpha(m)v) && \text{(left module structure of } V^*\text{)}\\
&= \left((\widehat{\alpha \otimes_{kG} (gf)})(m)\right)(v) && \text{(definition of } \widehat{\quad}\text{)}
\end{aligned}$$

Furthermore, the map $\widehat{\quad}$ is H-linear, for

$$(\widehat{h\alpha \otimes_{kG} f})(m) = f(h\alpha)(m) = f\alpha(h^{-1}m) = (\widehat{\alpha \otimes_{kG} f})(h^{-1}m) = (h(\widehat{\alpha \otimes_{kG} f}))(m).$$

Thus it remains to prove that $\widehat{\quad}$ is a k-isomorphism. First we consider the case $M = kG$. Let $\alpha \in Hom_{kG}(kG, V^*)$, hence

$$\alpha(\sum_{g \in G} a_g g) = \sum_{g \in G} a_g \alpha(g) = \sum_{g \in G} a_g \alpha(1) g = f \sum_{g \in G} a_g g$$

where $\alpha(1) = f \in V^*$ and $a_g \in k$ for $g \in G$. If id denotes the identity in $Hom_{kG}(kG, kG)$ then

$$(\widehat{id \otimes_{kG} f})(\sum_{g \in G} k_g g) = f \sum_{g \in G} k_g g = \alpha(\sum_{g \in G} k_g g),$$

hence $\widehat{id \otimes_{kG} f} = \alpha$. Thus the above map is an epimorphism, and therefore an isomorphism since the dimensions of the spaces in (ii) are obviously equal for $M = kG$. Since any free G-module is a direct sum of modules isomorphic kG the map $\widehat{\quad}$ is also an isomorphism if M is a free module. Since a projective module is a direct summand of a free module and since the isomorphisms are all compatible with taking direct summands the proof of (ii) is complete.

Claim (iii) $Hom_k(M, k) \cong Hom_{kG}(M, kG)$ as $H \times G$-bimodules:

Abstractly this follows by Frobenius reciprocity [9, Chapter VI, formula (8.7)] and the fact that kG is a symmetric algebra. However, we need the isomorphism explicit. Let $(\cdot, \cdot)$ be the symmetrizing bilinear form making kG into a symmetric algebra. For all $x, y \in kG$ the value of (x, y) is the coefficient of 1 in $xy \in kG$ (cf. e.g. [2, Theorem 62.1]). Then the map

$$\begin{array}{ccc} Hom_{kG}(M, kG) & \stackrel{\rho_M}{\longrightarrow} & Hom_k(M, k) \\ f & \mapsto & (m \mapsto (f(m), 1)) \end{array}$$

is a k-linear. If $\rho_M(f) = 0$ we get

$$0 = (f(mx), 1) = (f(m)x, 1) = (f(m), x)$$

for all $m \in M$ and all $x \in kG$. Since $(\cdot, \cdot)$ is non degenerate $f(m) = 0$ for all $m \in M$ and hence $f = 0$. Therefore ρ_M is injective.

Moreover, ρ_M is a morphism of $H \times G$-bimodules. Indeed, for $g \in G$ and $h \in H$, for $f \in Hom_{kG}(M, kG)$ and $m \in M$, we have

$$\begin{array}{rcll} (\rho_M(hfg))(m) & = & ((hfg)(m), 1) & \text{(definition of } \rho_M) \\ & = & (g^{-1} f(h^{-1}m), 1) & (H \times G\text{-bimodule structure of } Hom_{kG}(M, kG)) \\ & = & (1, g^{-1} f(h^{-1}m)) & \text{(since } (\cdot, \cdot) \text{ is symmetric)} \\ & = & (g^{-1}, f(h^{-1}m)) & \text{(since } (\cdot, \cdot) \text{ is associative)} \\ & = & (f(h^{-1}m), g^{-1}) & \text{(since } (\cdot, \cdot) \text{ is symmetric)} \\ & = & (f(h^{-1}m)g^{-1}, 1) & \text{(since } (\cdot, \cdot) \text{ is associative)} \\ & = & (f(h^{-1}mg^{-1}), 1) & (f \text{ is } kG\text{-linear}) \\ & = & \rho_M(f)(h^{-1}mg^{-1}) & \text{(definition of } \rho_M) \\ & = & (h(\rho_M(f))g)(m) & (H \times G\text{-bimodule structure of } Hom_k(M, k)) \end{array}$$

and so $\rho_M(hfg) = h(\rho_M(f))g$ which shows that ρ_M is $H \times G$-linear.

The inverse isomorphism is given by sending $\varphi \in Hom_k(M, k)$ to $\Lambda_M(\varphi) \in Hom_{kG}(M, kG)$ defined by

$$\Lambda_M(\varphi)(m) = \sum_{x \in G} \varphi(mx^{-1})x$$

for $m \in M$: First note, that $\Lambda_M(\varphi)$ is G-linear since it is the trace under the G-action. Furthermore,

$$\begin{array}{rcl} ((\rho_M \circ \Lambda_M)(\varphi))(m) & = & (\sum_{x \in G} \varphi(mx^{-1})x, 1) \\ & = & \sum_{x \in G} \varphi(mx^{-1})(x, 1) \\ & = & \sum_{x \in G} \varphi(mx^{-1})\delta_{x,1} \\ & = & \varphi(m) \end{array}$$

for $m \in M$ and $\varphi \in Hom_k(M, k)$ where δ denotes the Kronecker symbol. Thus, ρ_M is surjective and $\rho_M^{-1} = \Lambda_M$ is an isomorphism of $H \times G$-bimodules.

Since Λ_M is $H \times G$-linear, $\Lambda_M \otimes_{kG} id_{V^*}$ is well defined (cf. e.g. [2, Theorem 12.10]). Summarizing (i), (ii) and (iii) we get explicit isomorphisms

$$\begin{array}{rcl} M \otimes_{kG} V & \xrightarrow{\beta\otimes\alpha} & M^* \otimes_{kG} V^* \\ & \xrightarrow{\Lambda_M \otimes id_{V^*}} & Hom_{kG}(M, kG) \otimes_{kG} V^* \\ & \xrightarrow{\widehat{\quad}} & Hom_{kG}(M, V^*) \\ & \xrightarrow{\Phi} & Hom_k(M \otimes_{kG} V, k) = (M \otimes_{kG} V)^* \end{array}$$

as H-left modules.

We need to verify the formula given in the Proposition.

$$\begin{array}{rcl} m \otimes v & \overset{\beta\otimes\alpha}{\mapsto} & \beta(m) \otimes_{kG} \alpha(v) \\ & \overset{\Lambda_M\otimes id}{\mapsto} & \sum_{g\in G} \beta(mg^{-1})g \otimes_{kG} \alpha(v) \\ & = & \sum_{g\in G} \beta(mg^{-1}) \otimes_{kG} g\alpha(v) \\ & \overset{\widehat{\quad}}{\mapsto} & m' \mapsto \left(v' \mapsto \left(\sum_{g\in G} g\alpha(v)\left(\beta(mg^{-1})(m')v'\right)\right)\right) \\ & = & m' \mapsto \left(v' \mapsto \left(\sum_{g\in G} \beta(mg^{-1})(m') \cdot g\alpha(v)(v')\right)\right) \\ & \overset{\Phi}{\mapsto} & \left((m' \otimes v') \mapsto \left(\sum_{g\in G} \beta(mg^{-1})(m') \cdot \alpha(v)(g^{-1}v')\right)\right) \end{array}$$

This completes the proof. ■

The following is the translation of the isomorphism given in Proposition 3.3 in terms of bilinear forms.

Corollary 3.4. *Let M be an $H \times G$-bimodule which is projective as a G-module. Suppose that M carries a non-degenerate $H \times G$-invariant non-degenerate bilinear form b. Let V be a G-left module with a non-degenerate G-invariant bilinear form B. Then, for $m, m' \in M$ and $v, v' \in V$,*

$$\tilde{B}(m \otimes_{kG} v, m' \otimes_{kG} v') = \sum_{g\in G} b(mg^{-1}, m')B(v, g^{-1}v')$$

defines a non-degenerate H-invariant bilinear form on $M \otimes_{kG} V$.

Proof. By Lemma 3.2, the bilinear form b is associated to an $H \times G$-isomorphism $\beta : M \to M^*$ via

$$b(m, m') = \beta(m)m'$$

for $m, m' \in M$. Similarly, B is associated to a G-isomorphism $\alpha : V \to V^*$ via

$$B(v, v') = \alpha(v)v'$$

for $v, v' \in V$. Proposition 3.3 shows that an isomorphism

$$\Psi : M \otimes_{kG} V \longrightarrow (M \otimes_{kG} V)^*$$

is given by

$$\begin{aligned}(\Psi(m \otimes v))\,(m' \otimes v') &= \textstyle\sum_{g \in G} \beta(mg^{-1})(m') \cdot \alpha(v)(g^{-1}v') \\ &= \textstyle\sum_{g \in G} b(mg^{-1}, m')B(v, g^{-1}v').\end{aligned}$$

Thus, by Lemma 3.2, the bilinear form $\tilde{B}$ defined on $M \otimes_{kG} V$ by

$$\tilde{B}(m \otimes_{kG} v, m' \otimes_{kG} v') = \sum_{g \in G} b(mg^{-1}, m')B(v, g^{-1}v')$$

is non-degenerate and H-invariant. ∎

Theorem 3.5. *Let B_G and B_H be blocks for G and H respectively. Suppose that the $H \times G$-bimodule $M = {}_HM_G$ defines a Morita equivalence between B_G and B_H and suppose that M is self-dual. Then a B_G-left module V is self-dual if and only if $M \otimes_{kG} V$ is a self-dual B_H-left module.*

Proof. By Proposition 3.3, the functor $M \otimes_{kG} -$ maps self-dual G-modules to self-dual H-modules.

On the other hand, $Hom_{kH}(M, kH)$ becomes a $G \times H$-bimodule via

$$(g\alpha h)(m) = \alpha(mg)h$$

for all $g \in G, h \in H$ and $\alpha \in Hom_{kH}(M, kH)$. Then $Hom_{kH}(M, kH)$ is the inverse Morita bimodule to M and hence $Hom_{kH}(M, kH)$ defines the inverse functor from the category of B_H-modules onto the category of B_G-modules. In order to prove the only if part it is, by Proposition 3.3, enough to show that the $G \times H$-bimodule $Hom_{kH}(M, kH)$ is self-dual as well.

First note that

$$Hom_{kH}(M, kH) \cong Hom_k(M, k) = {}_G(M^*)_H$$

as $G \times H$-bimodules (by Claim (iii) in the proof of Proposition 3.3, changing left- and right structures there). If we define a $G \times H$-structure on M by setting

$$g \circ m \circ h = h^{-1}mg^{-1}$$

then the inverse of the given isomorphism ${}_HM_G \longrightarrow {}_HM_G^*$ defines an isomorphism from ${}_G(M^*)_H$ to $M \cong ({}_G(M^*)_H)^*$ and we are done. ∎

Example 2.3 shows that not all Morita bimodules do admit a non-degenerate invariant bilinear form.

Remark 3.6. Given two Morita equivalences between two blocks B_G and B_H of group algebras kG and kH, the composition of the one with the inverse of the other gives a Morita self-equivalence of the block B_G. Now, the isomorphism classes of Morita self-equivalences of B_G form the Picard group $Pic(B_G)$ which is isomorphic to the outer automorphism group $Out(B_G) := Aut_{k-\mathrm{alg}}(B_G)/Inn(B_G)$ where as usual $Inn(B_G)$ denotes the inner automorphisms of B_G. Each $\alpha \in Aut_{k-\mathrm{alg}}(B_G)$ induces an equivalence by sending a module V to ${}^{\alpha}V$, which is defined to be V as a k-vector space, but where the action of B_G on ${}^{\alpha}V$ is given by

$$a \bullet v = \alpha(a)v$$

for $a \in B_G$ and $v \in V$. Since $({}^{\alpha}V)^* \cong {}^{\alpha}(V^*)$ the automorphism α preserves the property of self-duality on modules. Since $B_H \simeq End_{B_G}(M)$, whatever isomorphism is taken to identify B_H with $End_{B_G}(M)$ the question whether a given module $M \otimes_{kG} V$ carries a non degenerate H-invariant bilinear form is independent of this choice.

There are Morita equivalences which are induced by self-dual bimodules.

Example 3.7. Let G be a finite p-solvable group and let $N = O_{p'}(G)$. We put $H = G/N$ and

$$e = \frac{1}{|O_{p'}(G)|} \sum_{g \in O_{p'}(G)} g.$$

Clearly, e is the block idempotent of the principal block B_0 of kG and $ekG \cong kG/N = kH$ as $H \times G$-bimodules where N obviously acts trivially on both sides. The bimodule ${}_H M_G = ekG$ is a Morita bimodule which induces an equivalence between the principal block of kG and the group algebra kH. In fact

$$End_{kG}(ekG) \cong ekGe \cong kG/N = kH,$$

and obviously ekG is a progenerator of $ekG = B_0$. Moreover, $ekG \cong (ekG)^*$ as $H \times G$-bimodules since, by a well-known result on duality,

$$(ekG)^* \cong \bar{e}kG = ekG$$

where $^-$ denotes the anti-automorphism of kG defined by $g \to \bar{g} = g^{-1}$. Thus the equivalence preserves self-duality, by Proposition 3.3. Clearly, this fact is obvious since N acts trivially on all modules in B_0.

4. Morita equivalence and forms in odd characteristic

Throughout this section let k be a field of odd characteristic (which includes the characteristic 0 case as well). A G-module V is called of *symmetric* resp. *antisymmetric type* if V carries a non-degenerate G-invariant symmetric resp. antisymmetric bilinear form.

Proposition 4.1. *Let M be an $H \times G$-bimodule inducing a Morita equivalence between a kG-block B_G and a kH-block B_H. Suppose that M is of symmetric type. Let V be a self-dual G-left module. If V is of symmetric resp. antisymmetric type then $M \otimes_{kG} V$ is of symmetric resp. antisymmetric type.*

Proof. Suppose b is a symmetric bilinear form and the bilinear form B is symmetric resp. antisymmetric if V is of symmetric resp. antisymmetric type. With the notation of Corollary 3.4, we have

$$\begin{aligned}
\tilde{B}(m \otimes_{kG} v, m' \otimes_{kG} v') &= \textstyle\sum_{g \in G} b(mg^{-1}, m')B(v, g^{-1}v') \\
&= \textstyle\sum_{g \in G} b(m, m'g)B(gv, v') \qquad \text{(since } b \text{ and } B \text{ are } G\text{-invariant)} \\
&= \pm \textstyle\sum_{g \in G} b(m'g, m)B(v', gv) \quad \text{(by symmetry of } b \text{ and } B\text{)} \\
&= \pm \tilde{B}(m' \otimes_{kG} v', m \otimes_{kG} v),
\end{aligned}$$

where the $+$ sign appears for symmetric and the $-$ sign for antisymmetric type modules V. ∎

Example 4.2. We consider again Example 3.7. Note that the Morita module ekG is of symmetric type. As a form one can take the standard non-degenerate G-invariant symmetric form on kG defined by

$$b(g, g') = \delta_{g,g'}$$

for $g, g' \in G$ restricted to ekG. The restriction is indeed non-degenerate as one can check easily. Thus Proposition 4.1 applies.

Now let V be a simple self-dual kG-module. Thus V carries a G-invariant non-degenerate bilinear form. If k is algebraically closed then - up to scalars - there is exactly one non-degenerate G-invariant form which is symmetric or antisymmetric since the characteristic is odd ([4],Chap. VII, 8.12). Thus, for k algebraically closed, a simple self-dual G-module is either of symmetric or antisymmetric type but not of both. In particular, the Morita module

$M = ekG$ in Example 4.2 is never of antisymmetric type since otherwise, by Proposition 4.1, there are simple H-modules which are of symmetric and antisymmetric type. Indeed it is well-known, that a group algebra $kH \cong ekG$ in odd characteristic never has a non-degenerate H-invariant antisymmetric form.

Theorem 4.3. *Let k be an algebraically closed field of odd characteristic. Let M be an $H \times G$-bimodule inducing a Morita equivalence between a kG-block B_G and a kH-block B_H. If M is of symmetric type then the equivalence preserves the type of simple self-dual modules.*

Proof. This is an immediate consequence of Proposition 4.1. ∎

Remark 4.4. Let k be an algebraically closed field of odd characteristic. By definition, a Morita bimodule $M = {}_HM_G$ is a projective kG-right and a projective kH-left module. If M carries a non-degenerate $H \times G$-invariant symmetric form then a projective indecomposable kG- resp. kH-module of antisymmetric type occurs with even multiplicity in M. Note that, by ([12], Proposition 2.2), an indecomposable projective module is of antisymmetric type if and only if its head is of antisymmetric type.

5. Morita equivalence and forms in even characteristic

In characteristic 2 the geometry of self-dual modules turns out to be more subtle. In this case an alternating form is symmetric. By Fong's Lemma ([4], Chap. VII, Theorem 8.13), a simple non-trivial self-dual kG-module V always carries a non-degenerate G-invariant symplectic form, but it may happen that V even has a non-degenerate G-invariant quadratic form. This happens if $H^1(G, V) = 0$ ([11], Proposition 2.4) and a connecting homomorphism to the first cohomology group carries the essential information what really happens. We call an arbitrary module $V \cong V^*$ of quadratic type if it carries a non-degenerate G-invariant quadratic form, say Q. The corresponding G-invariant non-degenerate symplectic form B defined by

$$B(v, v') = Q(v + v') - Q(v) - Q(v')$$

for $v, v' \in V$ is called the polarization of Q. Note that the quadratic form Q is non-degenerate if

$$\operatorname{rad} Q := \{v \mid v \in \operatorname{rad} B,\ Q(v) = 0\} = 0.$$

Throughout the following k is assumed to be a perfect field of characteristic 2. The next Lemma can be found as Proposition 2.2 in [3].

Lemma 5.1. *Let P be a projective kG-module not containing the projective cover of the trivial module. Then each G-invariant symmetric form on P is the polarization of a G-invariant quadratic form on P.*

If the Morita module M defines an equivalence and if M is of symmetric type then it preserves projective modules of quadratic type, more precisely

Theorem 5.2. *Let M be an $H \times G$-bimodule inducing a Morita equivalence between a kG-block B_G and a kH-block B_H. If M is of symmetric type then the equivalence preserves the property "being of quadratic type" on projective modules.*

Proof. Let P be a projective kG-module of quadratic type. Then P is an orthogonal sum of orthogonal indecomosable modules. By ([12], Lemma 3.6), an orthogonal indecomposable module is either indecomposable or of the form $S \oplus S^*$ where S is indecomposable. Since

$$(M \otimes_{kG} (S \oplus S^*))^* \cong M^* \otimes_{kG} (S \oplus S^*)^* \cong M \otimes_{kG} (S \oplus S^*)$$

by Claim (i), (ii) and (iii) in the proof of Proposition 3.3 we need to prove the Theorem only for orthogonal indecomposable projective modules.

So let P be an orthogonal indecomposable projective kG-module in B_G of quadratic type. Thus P carries a non-degenerate G-invariant symplectic form B which is also symmetric. According to Corollary 3.4 the form $\tilde{B}$ defined by

$$\tilde{B}(m \otimes_{kG} p, m' \otimes_{kG} p') = \sum_{g \in G} b(mg^{-1}, m')B(p, g^{-1}p')$$

for $m, m' \in M$ and $p, p' \in P$ is a non-degenerate H-invariant bilinear form on $M \otimes_{kG} P$. Since b and B are symmetric the form $\tilde{B}$ is symmetric as well. Thus, if the projective module $M \otimes_{kG} P$ does not contain the projective cover of the trivial kH-module, say $P_H(1)$, then $M \otimes_{kG} P$ is of quadratic type, by Lemma 5.1. In case it contains $P_H(1)$ it is either $P_H(1)$ or a direct sum of two copies of $P_H(1)$. But, by ([12], Remark 3.5 (b)), $P_H(1)$ and hence any finite direct sum of it is always of quadratic type if $2 \mid |H|$. In case $2 \nmid |H|$ the trivial module k is projective and obviously of quadratic type. This completes the proof. ∎

At the moment we are not able to prove an analogue of Theorem 5.2 for simple self-dual modules with the methods we have developed so far. However, it can be shown that a Morita equivalence between two principal 2-blocks which sends the trivial module to the trivial one preserves on simple modules the property "being of quadratic type" or not. This is a consequence of the theory of polynomial endo-functors of module categories of group rings.

References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker and R.A. Wilson ATLAS OF FINITE GROUPS, Clarendon Press, Oxford 1985.

[2] Charles W. Curtis and Irving Reiner, REPRESENTATION THEORY OF FINITE GROUPS AND ASSOCIATIVE ALGEBRAS, J. Wiley and Sons New-York 1966.

[3] Roderick Gow and Wolfgang Willems, *Quadratic geometries, projective modules, and idempotents*, Journal of Algebra **160** (1993), 257-272.

[4] Bertram Huppert and Norman Blackburn, FINITE GROUPS II, Springer Berlin 1982.

[5] Gerhard Hiss and Klaus Lux, BRAUER TREES OF SPORADIC GROUPS, Oxford Science Publishers 1989.

[6] Christoph Jansen, Klaus Lux, Richard Parker and Robert Wilson, AN ATLAS OF BRAUER CHARACTERS, Clarendon Press, Oxford 1995.

[7] Herbert Kupisch, *Projektive Moduln endlicher Gruppen mit zyklischer p-Sylow-Gruppe*, Journal of Algebra **10** (1968) 1-7.

[8] Herbert Kupisch, *Unzerlegbare Moduln endlicher Gruppen mit zyklischer p-Sylow-Gruppe*, Mathematische Zeitschrift **108** (1969) 77-104.

[9] Saunders MacLane, HOMOLOGY, Springer Verlag Berlin 1994.

[10] Irving Reiner, MAXIMAL ORDERS, Academic press 1975.

[11] Peter Sin and Wolfgang Willems, *G-invariant quadratic forms,* Journal für die Reine und Angewandte Mathematik **420** (1991), 45-59.

[12] Wolfgang Willems, *Duality and forms in representation theory*, in *Progress in Mathematics* Vol. **95**, Birkhäuser (1991), 509-520.

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