On the dimensions of PIM's

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Abstract We discuss the structure of finite groups for which the projective indecomposable modules have special given dimensions. In particular, we prove the converse of Fong's dimension formula for p-solvable groups. Furthermore, we characterize groups for which all irreducible p-Brauer characters have p-power degrees.

Keywords character degrees, principal indecomposable modules, p-solvable groups

1 Introduction

All groups in this note are assumed to be finite. The study of character degrees is a relevant subject in character theory. Many of the known results provide characterizations of groups for which a given property of their character degrees holds true. This approach has also been used in modular character theory, although in general the situation is more subtle than in the ordinary case.

Throughout this paper p is always a prime number. By $\mathrm{Irr}(G)$, respectively $\mathrm{IBr}_p(G)$ we denote the irreducible complex, respectively irreducible p-Brauer characters of the group G. To each $\varphi \in \mathrm{IBr}_p(G)$ we may associate the projective indecomposable module in characteristic p whose head affords φ as Brauer character (called the φ-PIM). We use the notation Φ_φ to denote its complex character. In particular, $\Phi_\varphi(1)$ is the dimension of the φ-PIM. In case φ is the trivial p-Brauer character we simply write Φ_1. Furthermore, we define (after Malle and Weigel, [13])

$$c_\varphi = \frac{\Phi_\varphi(1)}{|G|_p}$$

for $\varphi \in \mathrm{IBr}_p(G)$. In the following let P be a Sylow p-subgroup of G and let $\varphi^{\mathrm{Fix}(P)}(1)$ denote the dimension of the vector space of P-fixed points of the module afforded by φ. Obviously,

$$1 \le \varphi^{\mathrm{Fix}(P)}(1) \le (1, \Phi_\varphi|_P) = (1_P^G, \Phi_\varphi) = c_\varphi. \tag{1}$$

For c_φ, the following three situations are of particular interest.

$$\begin{array}{lll} \text{(i)} & c_\varphi = \varphi(1) & \text{for all } \varphi \in \mathrm{IBr}_p(G), \\ \text{(ii)} & c_\varphi = \varphi(1)_{p'} & \text{for all } \varphi \in \mathrm{IBr}_p(G), \\ \text{(iii)} & c_\varphi = \varphi^{\mathrm{Fix}(P)}(1) & \text{for all } \varphi \in \mathrm{IBr}_p(G), \\ \text{(iv)} & c_\varphi = 1 & \text{for all } \varphi \in \mathrm{IBr}_p(G). \end{array}$$

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The first case has already been considered by Brockhaus in [3]. He proved that $c_\varphi = \varphi(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$ if and only if the Sylow p-subgroup of G is normal. In [21] we conjectured

$$|G|_{p'} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2$$

with equality if and only if the Sylow p-subroup is normal. Unfortunately this conjecture is still open and we will discuss it slightly more in the last section of this paper.

Clearly, the second condition (ii) holds true for p-solvable groups, by Fong's dimension formula (see [16], Corollary 10.14). To be brief let $\mathrm{IBr}_p(\Phi_1)$ denote the set of irreducible p-Brauer characters which occur as constituents in Φ_1. We prove in section 2

Theorem A If the finite group G satisfies

$$c_\varphi = \varphi(1)_{p'} \quad \text{for all } \varphi \in \mathrm{IBr}_p(\Phi_1) \tag{2}$$

then G is p-solvable. In particular, condition (ii) characterizes p-solvable groups.

At a first glance condition (iii) does not seem to be very natural. However, as one of the main results we prove in section 3

Theorem B If the finite group G satisfies

$$c_\varphi = \varphi^{\mathrm{Fix}(P)}(1) \text{ for all } \varphi \in \mathrm{IBr}_p(\Phi_1) \tag{3}$$

then G is p-solvable. Moreover, G satisfies (3) for every $\varphi \in \mathrm{IBr}_p(G)$ if and only if the permutation module 1_P^G is completely reducible in characteristic p.

Finally, in case (iv), the condition $c_\varphi = 1$ implies $\varphi^{\mathrm{Fix}(P)}(1) = c_\varphi = 1$, by (1). Thus, if this condition holds for all $\varphi \in \mathrm{IBr}_p(G)$ then G is p-solvable according to Theorem B, and Fong's Dimension Formula yields

$$1 = c_\varphi = \varphi(1)_{p'}.$$

This means that all irreducible p-Brauer characters have p-power degrees. If p is odd then G has to be solvable, by Corollary 2 of [20]. Since a 2-solvable group is solvable by the Feit-Thompson Theorem, we see that G is a solvable group. Such groups with $c_\varphi = 1$ for all $\varphi \in \mathrm{IBr}_p(G)$ have been considered already in section 13 of [14]. In particular, in Corollary 13.10 it is proved that $G = \mathrm{O}_{p,p',p,p',p}(G)$. We give the full classification in section 4, namely

Theorem C The following conditions are equivalent.

a) $c_\varphi = 1$ for all $\varphi \in \mathrm{IBr}_p(G)$.

b) $G = \mathrm{O}_{p,p',p,p',p}(G)$ where $L = \mathrm{O}_{p,p',p,p'}(G)/\mathrm{O}_p(G)$ satisfies the following two conditions.

  α) All p'-sections of L are abelian.

  β) $L = C_L(x)\,\mathrm{O}_{p',p}(L)$ for all $x \in \mathrm{O}_{p'}(L)$.

In the literature there are several results which prove that properties of the classical irreducible constituents of Φ_1 heavily restrict the structure of the underlying group; see for instance [18], [15]. Theorem A and Theorem B show that properties on all irreducible Brauer characters occuring in Φ_1 also have a strong impact on the group structure.

2 The case $c_\varphi = \varphi(1)_{p'}$

We start this section with a useful lemma.

Lemma 2.1 *Let N be a normal subgroup of G.*

a) *If $\varphi \in \mathrm{IBr}_p(G)$ is obtained by inflation from $\bar\varphi \in \mathrm{IBr}_p(G/N)$ then $c_\varphi = c_{\bar\varphi}\, c_{1_N}$.*

b) *Let $\theta \in \mathrm{IBr}_p N$ and $\varphi \in \mathrm{IBr}_p G$ such that θ is a constituent of φ_N. Then*

$$c_\varphi = \begin{cases} \frac{\varphi(1)}{\theta(1)} c_\theta & \text{if } p \nmid |G:N| \\ c_\theta & \text{if } p = |G:N|. \end{cases}$$

Proof: a) By ([11], Chap. VII, 14.2), we have $\Phi_\varphi(1) = \Phi_{\overline{\varphi}}(1)\, \Phi_{1_N}(1)$ where Φ_{1_N} denotes the character of the 1-PIM of N.
b) Observe that the inertial groups $I = I_G(\theta)$ and $I_G(\Phi_\theta)$ coincide. If $p \nmid |G:N|$ then by ([16], Corollary 8.7 and Corollary 8.8), we immediately get $\Phi_\varphi(1) = \frac{\varphi(1)}{\theta(1)}\, \Phi_\theta(1)$. In case $|G:N| = p$ it suffices to note that Green's Indecomposability Theorem ([11], Chap. VII, 16.2) yields $\Phi_\theta^G = \Phi_\varphi$. $\square$

Lemma 2.2 *Let N be a normal subgroup of G. If G satisfies*

$$c_\varphi = \varphi(1)_{p'}$$

for all $\varphi \in \mathrm{IBr}_p(\Phi_1)$ then the factor group G/N satisfies the analogous equations. If moreover G/N is p-solvable, then so does N.

Proof: Note that the hypothesis implies $c_1 = c_{1_N} = 1$. Let $\Phi_{\overline{1}}$ denote the character of the 1-PIM of $\overline{G} = G/N$ and let $\overline{\varphi} \in \mathrm{IBr}_p(\overline{G})$ be in $\mathrm{IBr}_p(\Phi_{\overline{1}})$. If φ is the inflation of $\overline{\varphi}$ to G then obviously $\varphi \in \mathrm{IBr}(\Phi_1)$. Thus the first assertion follows by Lemma 2.1 a).

For the last part of the Lemma, let θ be a constituent of Φ_{1_N}. Since G/N is p-solvable by assumption, we may assume that either $p \nmid |G:N|$ or $|G:N| = p$. We consider first the case $p \nmid |G:N|$. As $\Phi_1|_N = \Phi_{1_N}$ there exists some $\varphi \in \mathrm{IBr}_p(\Phi_1)$ where θ is a constituent of φ_N (this also follows from Corollary 8.7 and Corollary 8.8 in [16], or from [11], Chap. VII, Lemma 14.2). According to the assumption of the Lemma we have $c_\varphi = \varphi(1)_{p'}$. Moreover, Dade's Theorem ([16] Theorem 8.30) says that $p \nmid \frac{\varphi(1)}{\theta(1)}$. Thus Lemma 2.1 b) yields $c_\theta = \theta(1)_{p'}$.

In case $|G:N| = p$ Green's Indecomposability Theorem ([11], Chap. VII, 16.2) yields $\Phi_{1_N}^G = \Phi_1$. Thus we may choose again a $\varphi \in \mathrm{IBr}_p(\Phi_1)$ where θ is a constituent of φ_N. In this case, either $\varphi(1) = \theta(1)$ or $\varphi(1) = p\theta(1)$, hence $\varphi(1)_{p'} = \theta(1)_{p'}$, and the assertion follows by Lemma 2.1 b). $\square$

Suppose that the next result has been proved already.

Proposition 2.3 *Let G be a simple non-abelian group. If*

$$c_\varphi = \varphi(1)_{p'}$$

for all $\varphi \in \mathrm{IBr}_p(\Phi_1)$ then G is a p'-group.

Proof of Theorem A Let G be a non p-solvable group of minimal order which satisfies the assumptions of Theorem A. By Proposition 2.3, the group G is not simple. Now we take any nontrivial normal subgroup N of G. By Lemma 2.2, the factor group G/N is p-solvable. Under this hypothesis Lemma 2.2 implies that N is p-solvable as well which completes the proof.

Thus we are left with the proof of Proposition 2.3. Let G be a simple non-abelian group satisfying

$$c_\varphi = \varphi(1)_{p'} \text{ for all } \varphi \in \mathrm{IBr}_p(\Phi_1). \tag{4}$$

If $p \mid |G|$ then, by Theorem A of [13], the group G belongs to one in the following list which we call the Malle-Weigel list in the sequel.

(a) A_p, $p \geq 5$

(b) $\mathrm{PSL}(2,p)$, $p \geq 5$

(c) $\mathrm{PSL}(n,q)$ where $\frac{q^n-1}{q-1} = p^f$ and $n \geq 3$

(d) $\mathrm{PSL}(2,q)$ where q is a Mersenne prime and $p = 2$

(e) $\mathrm{PSL}(2,q)$ where $p = q+1$ is a Fermat prime

(f) $\mathrm{PSL}(2,8)$ and $p = 3$

(g) M_{11}, $p = 11$

(h) M_{23}, $p = 23$

In each case we will find a $\varphi \in \mathrm{IBr}_p(\Phi_1)$ which does not satisfy the equation (4). This can be seen directly for $\mathrm{M}_{11}, \mathrm{M}_{23}$ and $\mathrm{PSL}(2,8)$ using the GAP library [7]. It also follows immediately in case (e) using ([2], case II) and in case (d) using ([2], case VIII (b)). Furthermore, $G = \mathrm{PSL}(2,p)$, $p \geq 5$ does not satisfies (4) either, since by ([1], Section 7), there is a $\varphi \in \mathrm{IBr}_p(\Phi_1)$ with $\varphi(1) = p-2 = \varphi(1)_{p'} > 2$ and

$$c_\varphi = \frac{\Phi_\varphi(1)}{p} = 2 \neq \varphi(1)_{p'}.$$

Thus we only have to deal with the cases in (a) and (c). But for further reference, we shall consider the whole list in the next result.

Lemma 2.4 *For each of the groups G in the Malle-Weigel list we have the following.*

a) *The Sylow p-subgroups of G are cyclic except in case* (d).

b) *Φ_1 contains exactly one non-trivial Brauer character β with $\beta(1) = |G|_p - 2 = \beta(1)_{p'}$ except possibly in case* (d).

Proof: a) This is clear for A_p and $\mathrm{PSL}(2,p)$. The condition $\frac{q^n-1}{q-1} = p^f$ for $n \geq 3$ in case (c) implies, by Zsigmondy's Theorem (see [11], Chap. IX, Theorem 8.3), that $|\mathrm{PSL}(n,q)|_p = p^f$ since $n \geq 3$ and $2^6 - 1 \neq p^f$. Moreover, in this case a Sylow p-subgroup is generated by a Singer cycle. Finally, for all groups in (e) - (h) the assertion is clear.
b) For A_p it follows by a result of Wielandt (see [11], Chap. XII, Theorem 10.7). In case (b) and (c) we can read it off from the Brauer trees given in ([1], p. 123) and ([5], Theorem C). Finally, case (e) and (f) follow by [2], case (g) and (h) by the GAP library [7]. □

Lemma 2.5 *Assume that $\Phi_1(1) = |G|_p$. If there is a $\beta \in \mathrm{IBr}_p(G)$ with $\beta \notin \mathrm{IBr}_p(\Phi_1)$ such that $\Phi_\beta \mid \Phi_1\Phi_1$, then for some $\varphi \in \mathrm{IBr}_p(\Phi_1)$ we have $c_\varphi < \varphi(1)$.*
In particular, (4) *does not hold true if in addition $p \nmid \varphi(1)$ for every $\varphi \in \mathrm{IBr}_p(\Phi_1)$.*

Proof: As for any $\varphi \in \mathrm{IBr}_p(G)$ the character Φ_φ is a constituent of the projective character $\varphi\Phi_1$, the first assumption implies $c_\varphi \leq \varphi(1)$. Suppose that $c_\varphi = \varphi(1)$ for all $\varphi \in \mathrm{IBr}_p(\Phi_1)$. Thus $\Phi_\varphi = \varphi\Phi_1$. So, if we put $\Phi_1 = \sum_{\varphi \in \mathrm{IBr}_p(\Phi_1)} a_\varphi \varphi$, then

$$\Phi_1\Phi_1 = \sum_{\varphi \in \mathrm{IBr}_p(\Phi_1)} a_\varphi \; \varphi\Phi_1 = \sum_{\varphi \in \mathrm{IBr}_p(\Phi_1)} a_\varphi \; \Phi_\varphi,$$

which contradicts the hypothesis. □

Note that the assertion of the previous Lemma holds true if $\Phi_1\Phi_1$ contains an irreducible character of p-defect zero.

Lemma 2.6 *Let $p \geq 5$ be a prime. For A_p the character $\Phi_1\Phi_1$ contains an irreducible constituent which belongs to a p-block of defect zero. In particular, A_p does not satisfy* (4).

Proof: Let $G = \mathrm{S}_p$ be the symmetric group on p letters and let $H = \mathrm{S}_{p-1}$. Recall that the classical irreducible characters of G are labeled by the partitions of p. By the hook length formula (see [6], 4.12), the irreducible character $\phi_{(p-2,2)}$ (corresponding to the partition $(p-2,2)$) belongs to a p-block of defect zero. Furthermore $1_H^G = \Phi_1$. Using Pieri's formula (see [6], part I, section 4) and Frobenius reciprocity we obtain

$$\begin{aligned} 1_H^G\, 1_H^G &= (1_H^G|_H)^G = ((\phi_{(p)} + \phi_{(p-1,1)})|_H)^G = (2\phi_{(p-1)} + \phi_{(p-2,1)})^G \\ &= 2\phi_{(p)} + 3\phi_{(p-1,1)} + \phi_{(p-2,2)} + \phi_{(p-2,1,1)}. \end{aligned}$$

Thus $\phi_{(p-2,2)}$ is a constituent of $\Phi_1\Phi_1$ and we may apply Lemma 2.5. Since $\Phi_1|_{\mathrm{A}_p}$ is equal to $\Phi_{1_{\mathrm{A}_p}}$ and $\phi_{(p-2,2)}$ restricts irreducibly to A_p the proof is complete according to Lemma 2.4 b). □

Lemma 2.7 *Let $N \leq M \leq G$ where $N, M \lhd G$ and $H = M/N$. Suppose that $p \nmid |N|\,|G/M|$. If $\Phi_{1_G}\Phi_{1_G}$ contains an irreducible character of p-defect zero then $\Phi_{1_H}\Phi_{1_H}$ does as well.*

Proof: According to ([11], Chap. VII, Lemma 14.2) we have $\Phi_{1_M} = \Phi_{1_G}|_M$. The same reference implies $\Phi_{1_H} = \Phi_{1_M}$. Finally observe that the irreducible constituents of the restriction of an irreducible character of p-defect zero to a normal subgroup are also of p-defect zero which completes the proof. □

Lemma 2.8 *Let $G = \mathrm{PSL}(n,q)$ with $p^f = (q^n-1)/(q-1)$ and $n \geq 3$. Then $\Phi_1\Phi_1$ has an irreducible constituent which belongs to a p-block of defect zero. In particular, G does not satisfy* (4).

Proof: By Lemma 2.7, we may assume that $G = \mathrm{GL}(n,q)$. As observed in the proof of Lemma 2.4, the condition $(q^n-1)/(q-1) = p^f$ implies $p^f = |G|_p$ (recall that $n \neq 2$). This means that the parabolic subgroup

$$H = \left\{ \begin{pmatrix} a & * \\ 0 & \mathrm{GL}(n-1,q) \end{pmatrix} \mid 0 \neq a \in \mathrm{GF}(q) \right\}$$

is a p-complement. Thus $\Phi_1 = 1_H^G = R_L^G(1)$ where L is the subparabolic subgroup of G isomorphic to $\mathrm{GL}(1,q) \times \mathrm{GL}(n-1,q)$ and R_L^G denotes the Harish-Chandra induction; i.e., $\mathrm{R}_L^G(\phi) = (\mathrm{infl}\,\phi)^G$ where $\mathrm{infl}\,\phi$ is the inflation of $\phi \in \mathrm{Irr}(L)$ to H. Observe also that the associated Weyl subgroup W_H is S_{n-1}. For any $x \in G \setminus H$ the Mackey formula yields

$$\Phi_1\Phi_1 = 1_H^G 1_H^G = ((1_H^G)|_H)^G = 1_H^G + 1_{H\cap H^x}^G$$

since the action of G on G/H is 2-transitive. If

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix}$$

then

$$H_1 = H \cap H^x = \left\{ \begin{pmatrix} a & 0 & * \\ 0 & b & * \\ 0 & 0 & \mathrm{GL}(n-2,q) \end{pmatrix} \mid 0 \neq a, b \in \mathrm{GF}(q) \right\}.$$

Assume first that $n > 3$. The group H_1 is contained in the parabolic subgroup associated to the partition $\lambda = (n-2, 2)$, i.e.,

$$H_1 \leq P_\lambda = \left\{ \begin{pmatrix} \mathrm{GL}(2,q) & * \\ 0 & \mathrm{GL}(n-2,q) \end{pmatrix} \right\}.$$

Thus $1_{H_1}^G = 1_{P_\lambda}^G + \rho$ for some character ρ. Let χ_λ be the unipotent character of G associated to the partition λ. As $1_{P_\lambda}^G = \mathrm{R}_{L_\lambda}^G(1)$ is also a character obtained by Harish-Chandra induction from the corresponding subparabolic subgroup L_λ we deduce, by ([5], Lemma A), that

$$(\chi_\lambda, 1_{P_\lambda}^G) = (\chi_\lambda, \mathrm{R}_{L_\lambda}^G(1)) = (\phi_\lambda, 1_{\mathrm{S}_\lambda}^{\mathrm{S}_n}) \neq 0$$

where $\phi_\lambda \in \mathrm{Irr}(\mathrm{S}_n)$ is the character corresponding to λ and $\mathrm{S}_\lambda \cong \mathrm{S}_2 \times S_{n-2}$ is the associated Young subgroup. Moreover, by the hook length formula for degrees of unipotent characters (see for example [8]), we have

$$\chi_\lambda(1) = q^d \frac{(q^n-1)(q^{n-3}-1)}{(q-1)(q^2-1)}$$

for a suitable d. The condition $(q^n-1)/(q-1) = p^f$ implies that n is odd. It follows $q^2 - 1 \mid q^{n-3} - 1$ for $n \geq 5$ and therefore $p^f \mid \chi_\lambda(1)$. Thus the character χ_λ belongs to a p-block of defect zero.

We are left with the case $n = 3$. We show again that $1_{H_1}^G$ contains an irreducible character of p-defect zero. The claim may be proved using the character table of $\mathrm{GL}(3,q)$, which was first computed by Steinberg in ([19], section 3). To simplify calculations one can proceed as follows: First check that

$$(1_{H_1}^G, \chi_{(2,1)})_G = (1_{H_1}, \chi_{(2,1)})_{H_1} = 3$$

and

$$(1_{H_1}^G, \chi_{(1,1,1)})_G = (1_{H_1}, \chi_{(1,1,1)})_{H_1} = 2.$$

As obviously $(1_{H_1}^G, 1_G)_G = 1$, and $1_{H_1}^G$ has degree $q(q+1)(q^2+q+1)$ we are left with a character of degree

$$q(q+1)(q^2+q+1) - 1 - 3q(q+1) - 2q^3 = (q^2-q-1)(q^2+q+1).$$

Moreover, the condition that q^2+q+1 is a prime power implies that $q \not\equiv 1 \bmod 3$ since otherwise $q^2+q+1 \equiv 0 \bmod 3$, hence $q^2+q+1 = 3^f$ and $3 \mid q-1$ which is impossible by Zsigmondy's Theorem. Now, consider any character of G of the form $\alpha\chi_\mu$ where $1 \neq \alpha$ is linear and χ_μ is unipotent. Since $Z = \mathrm{Z}(G) \leq H_1$ we have $Z \leq \operatorname{Ker}\chi_\mu$. Furthermore note that α is a character of the form $\alpha(x) = (\det x)^j$ for some j. So, if $3j \not\equiv 0 \bmod q-1$ then $\alpha|_Z \neq 1_Z$. Therefore, in this case, we have

$$(1_{H_1}^G, \alpha\chi_\mu)_G = (1_{H_1}, \alpha\chi_\mu)_{H_1} \leq (1_Z, \alpha\chi_\mu)_Z = \chi_\mu(1)(1_Z, \alpha)_Z = 0.$$

This implies that the irreducible constituents left of $1_{H_1}^G$ must have degree either $(q-1)^2(q+1)$ or a multiple of q^2+q+1 (see ([19] section 3). Note that $(q-1)^2(q+1) \equiv 3 \bmod q^2+q+1$. So if all the rest irreducible components of $1_{H_1}^G$ were of degree $(q-1)^2(q+1)$, as as q^2+q+1 is a prime power we would deduce that q^2-q-1 is a multiple of $(q-1)^2(q+1)$ which is impossible. Thus $1_{H_1}^G$ and therefore $\Phi_1\Phi_1$ must contain some irreducible character of degree a multiple of q^2+q+1. All such characters belong to p-blocks of defect zero. Thus we have shown that in all cases $\Phi_1\Phi_1$ contains an irreducible character of p-defect zero. Again, the result follows by Lemma 2.4 b). $\square$

3 The case $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$

Let F denote the underlying field of characteristic p (always large enough) and let P be a Sylow p-subgroup of G. By abuse of notation we denote by 1_P the trivial FP-module (and also its ordinary character). For $\varphi \in \mathrm{IBr}_p(G)$ and V the FG-module affording φ, we put

$$\varphi^{\mathrm{Fix}(P)}(1) = \dim_F \{v \in V \mid vx = v \text{ for all } x \in P\} = \dim_F \operatorname{Hom}_{FP}(1_P, V).$$

Observe that this definition does not depend on the chosen p-Sylow subgroup. The second part of Theorem B is easy to see.

Proposition 3.1 *The following conditions are equivalent.*

a) $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$ *for all* $\varphi \in \mathrm{IBr}_p(G)$.

b) 1_P^G *is completely reducible.*

Proof: Observe that the multiplicity of φ in 1_P^G is equal to

$$(1_P^G, \Phi_\varphi) = (1_P, \Phi_\varphi|_P) = c_\varphi.$$

On the other hand, the degree $\varphi^{\mathrm{Fix}(P)}(1)$ equals to the multiplicity of φ in the head of 1_P^G, by Nakayama's Theorem. Thus $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$ if and only if 1_P^G is completely reducible. $\square$

We may naturally ask: Which are exactly the groups that satisfy $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$?

Lemma 3.2 *Let G be a group and let N be a normal subgroup of G.*

a) *If 1_P^G is completely reducible then $1_{PN/N}^{G/N}$ and $1_{P\cap N}^N$ are completely reducible as well.*

b) *The converse of a) holds true if N is a p-group or $p \nmid |G/N|$.*

Proof: a) Since $1_{PN/N}^{G/N}$ is a factor module of 1_P^G and $1_{P\cap N}^N$ is a direct summand of the restriction $1_P^G|_N$ the statement follows.
b) If N is a normal p-subgroup we have $1_P^G \cong 1_{P/N}^{G/N}$. If p does not divide $|G/N|$ the assertion is a consequence of the fact that an FG-module U is completely reducible if and only if the restriction U_N is completely reducible (see [11], Chap. VII, Theorem 7.21). □

Lemma 3.3 *Let N be a normal subgroup of G.*

a) *If $\varphi \in \mathrm{IBr}_p(G)$ is obtained by inflation from $\bar{\varphi} \in \mathrm{IBr}_p(G/N)$ then*

$$\varphi^{\mathrm{Fix}(P)}(1) = \bar{\varphi}^{\mathrm{Fix}(PN/N)}(1).$$

b) *Let $\varphi \in \mathrm{IBr}_p(G)$. If $\theta \in \mathrm{IBr}_p(N)$ is a constituent of φ_N then*

$$\varphi^{\mathrm{Fix}(P)}(1) = \tfrac{\varphi(1)}{\theta(1)}\theta^{\mathrm{Fix}(P)}(1) \qquad \text{if } p \nmid |G:N|$$

and

$$\varphi^{\mathrm{Fix}(P)}(1) \leq \theta^{\mathrm{Fix}(P\cap N)}(1) \qquad \text{if } p = |G:N|.$$

Proof: a) This is obvious.
b) Let V and W be modules over F affording φ and θ respectively. We first consider the case $p \nmid |G:N|$, hence $P \leq N$. Note that any G-conjugate character θ^x of θ satisfies $(\theta^x)^{\mathrm{Fix}(P)}(1) = \theta^{\mathrm{Fix}(P)}(1)$. Therefore, by Nakayama's Lemma,

$$\varphi^{\mathrm{Fix}(P)}(1) = \dim_F \mathrm{Hom}_{FG}(1_P^G, V) = \dim_F \mathrm{Hom}_{FN}(1_P^N, V_N) = \frac{\varphi(1)}{\theta(1)}\theta^{\mathrm{Fix}(P)}(1).$$

In case $p = |G:N|$ we have $G = PN$. Thus, by Nakayama's Lemma and Mackey's Formula, we have

$$\varphi^{\mathrm{Fix}(P)}(1) \leq \dim_F \mathrm{Hom}_{FG}(1_P^G, W^G) = \dim_F \mathrm{Hom}_{FN}(1_{P\cap N}^N, W) = \theta^{\mathrm{Fix}(P\cap N)}(1).$$

□

The next result proves the first part of Theorem B.

Theorem 3.4 *If $\varphi^{\mathrm{Fix}(P)}(1) = c_\varphi$ for every $\varphi \in \mathrm{IBr}_p(\Phi_1)$ then G is p-solvable.*

Proof: Let G be a minimal counterexample and note hat $c_1 = 1^{\mathrm{Fix}(P)}(1) = 1$. We assume first that G is not simple. Let $1 \neq N \lhd G$. Then Lemma 2.1 a) together with Lemma 3.3 a) imply that G/N satisfies the same hypothesis as G. Thus we may assume by induction that G/N is p-solvable. In particular, we may assume that either $p \nmid |G:N|$ or $p = |G:N|$. If we prove that N satisfies the hypothesis as well then we are done by induction. Let $\theta \in \mathrm{IBr}_p(\Phi_{1_N})$. By the same argument as in Lemma 2.2 we see in both cases that there is some $\varphi \in \mathrm{IBr}_p(\Phi_1)$ such that θ is a constituent of φ_N. The hypothesis of the Theorem says $\varphi^{\mathrm{Fix}(P)}(1) = c_\varphi$. If $p \nmid |G:N|$ then Lemma 2.1 b) together with Lemma 3.3 b) imply the same for θ. In case $|G:N| = p$, Lemma 3.3 b) yields

$$c_\varphi = \varphi^{\mathrm{Fix}(P)}(1) \leq \theta^{\mathrm{Fix}(P\cap N)}(1) \leq c_\theta.$$

According to Lemma 2.1 b) we have $c_\varphi = c_\theta$ and we are done.

So we may assume that G is a simple non-abelian group of order divisible by p. Then G is one of the groups in Malle-Weigel's list of Section 3.

First we consider any group in this list but not being in (d). By Lemma 2.4, the Sylow p-subgroups of G are cyclic. Let Q be an FG-module affording Φ_1. Clearly, $Q|_P = FP$ since $\Phi_1(1) = |P|$. Furthermore, $Q|_P$ is uniserial since P is cyclic. It follows that the module affording the only non-trivial irreducible character $\beta \in \Phi_1$ (see Lemma 2.4) is uniserial as well when restricted to P which proves that $c_\beta = \beta^{\mathrm{Fix}(P)}(1) = 1$. Since $\beta(1) = |P| - 2$ we obtain

$$|P| = \Phi_\beta(1) = 2\beta(1) + m = 2|P| - 4 + m$$

where $m > 1$ is the degree of some Brauer character. Thus $|P| = 4 - m \leq 2$, which is impossible.

It remains to deal with the case $G = \mathrm{PSL}(2, q)$ where q is a Mersenne prime and $p = 2$. In particular, $q + 1 = 2^t$ for some $t \geq 2$. If Q denotes again the FG-module affording Φ_1 then the heart $\mathrm{H}(Q)$ of Q satisfies $\mathrm{H}(Q) = V \oplus W$ with irreducible modules V, W, both of dimension $\frac{2^t-2}{2} = 2^{t-1} - 1$ (see [4], Theorem 4). Since $Q|_P = FP$ and $\mathrm{H}(FP)$ is a direct sum of two uniserial modules (note that P is a dihedral group) we see that both modules V and W have a one-dimensional fixed point subspace for P. Thus, by the hypothesis of the Theorem, $\Phi_\varphi(1) = \Phi_\beta(1) = |G|_2 = 2^t$ where φ and β are the Brauer characters of V and W, respectively. Furthermore, since $\Phi_\varphi(1) \geq 2\varphi(1) = 2^t - 2$ we see that Φ_φ contains 1_G exactly with multiplicity 2. But, by symmetry of the Cartan matrix, this multiplicity equals the multiplicity of φ in Φ_1 which is 1. Thus we have a contradiction which completes the proof. $\square$

Note that according to Proposition 3.1 the previous result implies that G is p-solvable if 1_P^G is completely reducible.

Proposition 3.5 *Suppose that* $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$ *for all* $\varphi \in \mathrm{IBr}_p(G)$. *If* $H = \mathrm{O}^{p'}(G)$ *then*

a) $p \mid \varphi(1)$ *for all* $1 \neq \varphi \in \mathrm{IBr}_p(H)$.

b) $\mathrm{N}_H(P) = P$.

c) H *is solvable.*

Proof: The hypothesis of the Proposition on G passes on to H, by Proposition 3.1 and Lemma 3.2 b). By Theorem 3.4, the group H is p-solvable, and by Lemma 3.2, the module 1_P^H is completely reducible.
a) Suppose there is some $1 \neq \varphi \in \mathrm{IBr}_p(H)$ with $p \nmid \varphi(1)$. Since H is p-solvable we obtain

$$\varphi(1) = \varphi(1)_{p'} = c_\varphi = \varphi^{\mathrm{Fix}(P)}(1).$$

Thus P is contained in the kernel of φ which is therefore a proper normal subgroup of H of index prime to p. This contradicts the fact that $H = \mathrm{O}^{p'}(G)$.

b) In a p-solvable group X the number of irreducible p-Brauer characters of p'-degree equals the number of irreducible classical characters of $\mathrm{N}_X(P)/P$ where P is a Sylow p-subgroup of X ([22], Theorem A). Thus by a) we get $\mathrm{N}_H(P) = P$.
c) This follows immediately from Theorem 1.1 of [10] since H is p-solvable. $\square$

We would like to mention here that recently (see [17]) it has been proved that for a general finite group G and p an odd prime the conditions a) and b) in Proposition 3.5 are equivalent.

Let G be a p-solvable group and let $\varphi \in \mathrm{IBr}_p(G)$. We say that a pair pair (J, γ) where $J = J_\varphi \leq G$ and $\gamma = \gamma_\varphi \in \mathrm{IBr}_p(J)$ is a Huppert pair if γ is of p'-degree and $\gamma^G = \varphi$. By a result of Huppert ([16], Theorem 10.11), every $\varphi \in \mathrm{IBr}_p(G)$ has a Huppert pair.

Proposition 3.6 *Let G be a p-solvable group and let $\varphi \in \mathrm{IBr}_p(G)$ with Huppert pair (J, γ). Then the following conditions are equivalent.*

a) $\varphi^{\mathrm{Fix}(P)}(1) = c_\varphi = \varphi(1)_{p'}$.

b) $p \nmid |J : \mathrm{Ker}\ \gamma|$ *and* $P^x \cap J \in \mathrm{Syl}_p(J)$ *for all* $x \in G$.

Proof: Assume that part a) holds. Let V denote the FG-module affording φ and let W be the FJ-module affording γ. Since $W^G = V$ Mackey's Formula and Nakayama's Lemma yield

$$\begin{aligned}
\varphi^{\mathrm{Fix}(P)}(1) &= \dim \mathrm{Hom}_{FP}(1_P, V_P) = \dim \mathrm{Hom}_{FG}(1_P^G, V) \\
&= \dim \mathrm{Hom}_{FG}(1_P^G, W^G) = \dim \mathrm{Hom}_{FJ}(1_P^G|_J, W) \\
&= \textstyle\sum_{x \in P\backslash G/J} \dim \mathrm{Hom}_{FJ}(1_{P^x\cap J}^J, W) \\
&\leq \textstyle\sum_{x \in P\backslash G/J} (1_{P^x\cap J}^J, \Phi_\gamma) = \sum_{x \in P\backslash G/J} \Phi_\gamma(1) \frac{|J:P^x\cap J|}{|J|} \\
&= \tfrac{1}{|J|} \Phi_\gamma(1) \textstyle\sum_{x \in P\backslash G/J} |J : P^x \cap J| = \\
&= \tfrac{1}{|J|} \gamma(1) |J|_p |G : P| = |G : J|_{p'} \gamma(1) = \varphi(1)_{p'}.
\end{aligned}$$

Thus the condition $\varphi^{\mathrm{Fix}(P)}(1) = \varphi(1)_{p'}$ is equivalent to

$$\dim \mathrm{Hom}_{FJ}(1_{P^x\cap J}^J, W) = \gamma(1) \frac{|J|_p}{|P^x \cap J|} \text{ for every } x. \tag{5}$$

Observe that we may assume $P \cap J \in \mathrm{Syl}_p(J)$. Therefore (5) implies

$$\gamma^{\mathrm{Fix}(P)}(1) = \dim \mathrm{Hom}_{FJ}(1_{P\cap J}^J, W) = \gamma(1),$$

which means that $P \cap J$ acts trivially on W. Thus $\mathrm{Ker}\ \gamma \trianglelefteq J$ contains all the p-subgroups of J. In turn, this implies that $P^x \cap J \leq \mathrm{Ker}\ \gamma$ for every $x \in G$. It follows

$$\gamma(1) = \dim \mathrm{Hom}_{FJ}(1_{P^x\cap J}^J, W) = \gamma(1) \frac{|J|_p}{|P^x \cap J|},$$

hence $|P^x \cap J| = |J|_p$. Thus $P \cap J$ is a Sylow p-subgroup of J for every Sylow p-subgroup P of G.

Conversely, suppose that part b) holds. If $p \nmid |J : \operatorname{Ker} \gamma|$ then for every $x \in G$ we have

$$\gamma(1) = \dim \operatorname{Hom}_{FJ}(1_{P^x \cap J}^J, W),$$

and if moreover $|J|_p = |P^x \cap J|$, we obtain (5) which is equivalent to the statement in a). $\square$

As a consequence we have the next proposition. Recall that if G is p-solvable and K is a p-complement, a Fong character associated to $\varphi \in \operatorname{IBr}_p G$ is a $\chi \in \operatorname{Irr}(K)$ such that $\chi^G = \Phi_\phi$. By ([16] Theorem 10.13), any $\varphi \in \operatorname{IBr}_p(G)$ has some associated Fong character.

Proposition 3.7 *Assume that G has a normal p-complement K. Let $\varphi \in \operatorname{IBr}_p(G)$ and let $\chi \in \operatorname{Irr}(K)$ be an associated Fong character. Then $\varphi^{\operatorname{Fix}(P)}(1) = \varphi(1)_{p'}$ if and only if the inertial group $\mathrm{I}_P(\chi)$ of χ in P acts trivially on $K/\operatorname{Ker}\chi$.*

Proof: First observe that the condition on $\mathrm{I}_P(\chi)$ does not depend on the chosen Sylow p-subgroup. Moreover, $\mathrm{I}_{P^x}(\chi)$ is a Sylow p-subgroup of $J = \mathrm{I}_G(\chi) = (J \cap P)K = \mathrm{I}_{P^x}(\chi)K$ for every $x \in G$. Furthermore $\mathrm{I}_P(\chi)\operatorname{Ker}\chi \trianglelefteq J = \mathrm{I}_P(\chi)K$ if and only if

$$[\mathrm{I}_P(\chi), K] \leq \mathrm{I}_P(\chi)\operatorname{Ker}\chi \cap K = \operatorname{Ker}\chi$$

which is equivalent to the fact that $\mathrm{I}_P(\chi)$ acts trivially on $K/\operatorname{Ker}\chi$.

Assume first that $\varphi^{\operatorname{Fix}(P)}(1) = \varphi(1)_{p'}$. By a result of Dade (see [16], Theorem 8.13), there exists a $\gamma \in \operatorname{IBr}_p(J)$ such that $\chi = \gamma|_K$. In particular, γ has p'-degree. Thus (J, γ) is a Huppert pair for φ, and by Proposition 3.6, we obtain $p \nmid |J : \operatorname{Ker}\gamma|$. Hence $\mathrm{I}_P(\chi) \leq \operatorname{Ker}\gamma$. Since $\operatorname{Ker}\gamma \cap K = \operatorname{Ker}\chi$ and $J = I_P(\chi)K$ we get $\mathrm{I}_P(\chi)\operatorname{Ker}\chi = \operatorname{Ker}\gamma \trianglelefteq J$.

Conversely, suppose that $T = \mathrm{I}_P(\chi)\operatorname{Ker}\chi \trianglelefteq J$. Since $J/T \cong K/\operatorname{Ker}\chi$ we may consider the Brauer character γ of J obtained from χ by inflation. It follows again that (J, γ) is a Huppert pair for φ and since $\operatorname{Ker}\gamma = T$, we get $p \nmid |J : \operatorname{Ker}\gamma|$. Recall that for every $x \in G$ the intersection $P^x \cap J$ is a Sylow p-subgroup of J. Thus condition b) in Proposition 3.6 is satisfied from which $\varphi^{\operatorname{Fix}(P)}(1) = \varphi(1)_{p'}$ follows. $\square$

Corollary 3.8 *Assume that G has a normal p-complement K. Then 1_P^G is completely reducible if and only if $\mathrm{I}_P(\chi)$ acts trivially on $K/\operatorname{Ker}\chi$ for all $\chi \in \operatorname{Irr}(K)$.*

Proof: Since G has a normal p-complement Green's Indecomposability Theorem ([11], Chap. VII, 16.2) yields that all irreducible characters of K are Fong characters. Thus the statement follows immediately by Proposition 3.7 and Proposition 3.1. $\square$

Example 3.9 Let $F = \mathrm{GF}(3)$ and $p = 2$. We put

$$K = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \mid * \in F \right\}$$

and

$$P = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \mid 0 \neq a, b, c \in F \right\}.$$

Thus $G = KP$ is the Borel subgroup of $\mathrm{GL}(3,3)$ and $K \trianglelefteq G$. One easily checks that $\mathrm{O}^{2'}(G) = G$ and $\mathrm{N}_G(P) = P$. From the characters of the extraspecial normal subgroup K we can construct all $\varphi \in \mathrm{IBr}_2(G)$. There is exactly one φ of degree 6 and all other Brauer characters $\varphi \neq 1$ are nonlinear and have 2-power degree. We consider the Brauer character φ of degree 6. The two faithful characters, say χ and χ' of K of degree 3 are Fong characters for φ and $\mathrm{C}_P(K) = 1$. On the other hand $|\mathrm{I}_P(\chi)| = 4$. Therefore 1_P^G is not completely reducible by Corollary 3.8. The example shows that the conditions a) - c) do not characterize groups for which 1_P^G is completely reducible.

Corollary 3.10 *Assume that G has a normal p-complement K. Then 1_P^G is completely reducible if for every $g \in P$ we have $\mathrm{C}_{L_g}(g) = 1$, where*

$$L_g = \cap\{T \trianglelefteq K \mid g \in \mathrm{N}_P(T), \text{ and } g \text{ acts trivially on } K/T\}.$$

Proof: Let $g \in P$ and $\chi \in \mathrm{Irr}(K)$ with $g \in \mathrm{I}_P(\chi)$. Let $\phi \in \mathrm{Irr}(L_g)$ such that $\chi|_{L_g}$ contains ϕ as a constituent. As $C_{L_g}(g) = 1$ and L_g has p'-order, one easily sees that there is no nontrivial L_g-conjugacy class in L_g which is setwise fixed under the action of g. Thus if $1 \neq \phi \in \mathrm{Irr}(L_g)$, we have $\phi^g \neq \phi$. Since g permutes the irreducible constituents of $\chi|_{L_g}$ and as the number of those is prime to p, we deduce that some is fixed. This forces $\phi = 1_N$. Thus $L_g \leq \mathrm{Ker}\chi$ and g acts trivially on $K/\mathrm{Ker}\,\chi$. This shows that $\mathrm{I}_P(\chi)$ acts trivially on $K/\mathrm{Ker}\,\chi$, and we may apply Corollary 3.8. □

Lemma 3.11 *Let G be a group with normal p-complement K and let $H = O^{p'}(G)$. If*

$$L = \cap\{T \trianglelefteq K \mid P \leq \mathrm{N}_G(T), \text{ and } P \text{ acts trivially on } K/T\}$$

then $\mathrm{N}_H(P) = P$ is equivalent to $\mathrm{C}_L(P) = 1$.

Proof: Let $S = H \cap K$, hence $H = PS$. We shall prove below that $L = S$. From this we see that $\mathrm{N}_H(P) = P$ is equivalent to $\mathrm{C}_L(P) = \mathrm{C}_S(P) = 1$.

Observe that $L \leq S$ is a consequence of the fact that S is normal in G and that $G/S \cong H/S \times K/S \cong P \times K/S$. Thus P acts trivially on K/S.

In order to prove $S \leq L$, note that P acts trivially on K/L. Therefore PL is normalized by K. Thus PL is normal in G. Consequently $H \leq PL$ and therefore $S = H \cap K \leq PL \cap K = L$ which completes the proof. □

4 The case $c_\varphi = 1$

In the introduction we mentioned already that the condition $c_\varphi = 1$ for all $\varphi \in \mathrm{IBr}_p(G)$ implies that G is solvable and all irreducible p-Brauer characters have p-power degrees. The latter holds true, for instance, if G has an abelian p-complement by ([16], Theorem 10.13), since in this case all Fong characters are linear, or if all p'-conjugacy classes have p-power size ([20], Corollary 4).

Lemma 4.1 *Let $G = \mathrm{O}_{p',p,p'}(G)$. Then $\varphi(1)$ is a power of p for every $\varphi \in IBr_p(G)$ if and only if the following two conditions are satisfied.*

a) *All p'-sections of G are abelian.*

b) $G = \mathrm{C}_G(x)\,\mathrm{O}_{p',p}(G)$ *for all* $x \in \mathrm{O}_{p'}(G)$.

Proof: First suppose that all irreducible p-Brauer character of G have p-power degree. Since this condition passes on to normal subgroups and quotients we obviously have a).

Let $N = \mathrm{O}_{p',p}(G)$. If $\phi \in \mathrm{IBr}_p(N)$ then $\mathrm{I}_G(\phi) = G$ by Clifford, since $p \nmid |G : N|$. By Brauer's Permutation Lemma for irreducible Brauer characters (see [9] Theorem 7.5) the group G acts trivially on the N-conjugacy classes of p'-elements of N. Now take any $g \in G$ and $x \in O_{p'}(G)$. As $x^g \in O_{p'}(G) \leq N$ we get $x^g = x^n$ for some $n \in N$. Thus $gn^{-1} \in C_G(x)$ and part b) follows.

Conversely, assume that a) and b) hold. By part a) and ([16], Theorem 8.30) we get that all $\phi \in \mathrm{IBr}_p(N)$ have p-power degrees. Furthermore, part b) together with Brauer's Permutation Lemma imply $\mathrm{I}_G(\phi) = G$ for all $\phi \in \mathrm{IBr}_p(N)$. Thus we only have to check that any such $\phi \in \mathrm{IBr}_p(N)$ can be extended to G. Since $|G/N|$ and $\phi(1)$ are coprime, by ([16], Theorem 8.23) it suffices to prove that p does not divide the order of $\det \phi$. But this is obvious since $\det \phi$ is a linear Brauer character. $\square$

The next result contains in particular the statement of Theorem C.

Theorem 4.2 *The following conditions are equivalent.*

a) $c_\varphi = 1$ *for any* $\varphi \in \mathrm{IBr}_p(G)$,

b) $\varphi(1)$ *is a power of p for every* $\varphi \in \mathrm{IBr}_p(G)$, *and G is solvable if* $p = 2$.

c) $G = O_{p,p',p,p',p}(G)$ *and* $L = O_{p,p',p,p'}(G)/O_p(G)$ *satisfies* a) *and* b) *in* Lemma 4.1.

Proof: We first claim that a) and b) are equivalent. Since $1 \leq \varphi^{\mathrm{Fix}(P)}(1) \leq c_\varphi$ part a) together with Theorem B imply that G is p-solvable.

Hence, by the Feit-Thompson Theorem, G is solvable in case $p = 2$. Furthermore $1 = c_\varphi = \varphi(1)_{p'}$ for all $\varphi \in \mathrm{IBr}_p(G)$. Thus we have b). On the other hand, if b) holds then by ([20], Corollary 2), the group G is solvable. Hence $c_\varphi = \varphi(1)_{p'} = 1$ for every $\varphi \in \mathrm{IBr}_p(G)$ and we have a).

Now we prove that b) implies c). Recall that as observed before G is solvable. Thus Theorem 13.10 of [14] implies $G = O_{p,p',p,p',p}(G)$. Since our hypothesis passes on to sections, L is characterized by a) and b) of Lemma 4.1.

Finally, c) means, using again Lemma 4.1, that all irreducible p-Brauer characters of L have degree a power of p. Obviously, the same happens for $T = O_{p,p',p,p'}(G)$ since normal p-subgroups are in the kernel of irreducible representations in characteristic p. Thus, by ([16] Theorem 8.30), we get b). $\square$

Remark 4.3 Whereas in Sections 2 and 3 we have considered only irreducible Brauer characters in $\mathrm{IBr}_p(\Phi_1)$, we are dealing in this section with the whole character set $\mathrm{IBr}_p(G)$. Using a proof along the lines of the proof of Theorem B, one sees that if $c_\varphi = 1$ for any $\varphi \in \mathrm{IBr}_p(\Phi_1)$, then G must be p-solvable. But any group with a normal p-complement shows that the full characterization in Theorem C doesn't work if one considers only characters in $\mathrm{IBr}_p(\Phi_1)$. However, it might be true that there were some restriction on the p-length.

Question 4.4 *Let G be a group with $c_\varphi = 1$ for any $\varphi \in \mathrm{IBr}_p(\Phi_1)$. As remarked above G is p-solvable. Is there some bound on the p-length of such a group G?*

5 The case $c_\varphi = \varphi(1)$

Not much is known about c_φ for groups which are not p-solvable. For example, if $c_1 = 1$ then $c_\varphi \leq \varphi(1)$, but this does not hold true in general. However, in the average it might be true as many examples show. To be more precise, let $\mathrm{IBr}_p(G) = \{\varphi_1, \ldots, \varphi_k\}$. We denote by $\overline{\varphi}$ the dimension vector $(\varphi_1(1), \ldots, \varphi_k(1))$ and by c the vector $(c_{\varphi_1}, \ldots, c_{\varphi_k})$.

Furthermore, let $< \cdot, \cdot >$ denote the euclidean scalar product on $\mathbb{R}^k$ with corresponding norm $\| \cdot \|$. In particular, $\| \overline{\varphi} \|^2 = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2$. With this notation we state

Conjecture 5.1 *The inequality*

$$\| c \| \leq \| \overline{\varphi} \|$$

holds true for every finite group.

Based on many examples we conjectured in [21] the following.

Conjecture 5.2 *We always have*

$$|G|_{p'} \leq \| \overline{\varphi} \|^2 \tag{6}$$

with equality if and only if the Sylow p-subgroup is normal.

Proposition 5.3 *If Conjecture* 5.1 *has an affirmative answer then Conjecture* 5.2 *has an affirmative answer including the characterization of equality.*

Proof: Conjecture 5.1 states that

$$< c, c > \leq < \overline{\varphi}, \overline{\varphi} > .$$

Applying the Cauchy-Schwarz inequality we get

$$< c, \overline{\varphi} >^2 \leq < c, c > < \overline{\varphi}, \overline{\varphi} > \leq < \overline{\varphi}, \overline{\varphi} >^2 . \tag{7}$$

This proves that

$$(|G|_{p'})^2 = \frac{1}{(|G|_p)^2} (\sum_{i=1}^{k} \Phi_{\varphi_i}(1)\varphi_i(1))^2 \ \leq \ (\sum_{i=1}^{k} \varphi_i(1)^2)^2 = \| \overline{\varphi} \|^4,$$

which is the first part of Conjecture 5.2.

Next we characterize equality. Clearly, if the Sylow p-subgroup is normal then $p \nmid \varphi_i(1)$ for all i and by Fong's Theorem, $\Phi_{\varphi_i}(1) = |G|_p \, \varphi_i(1)$. Thus we get

$$|G|_{p'} = \frac{1}{|G|_p} (\sum_{i=1}^{k} \Phi_{\varphi_i}(1)\varphi_i(1)) = \sum_{i=1}^{k} \varphi_i(1)^2 = \| \overline{\varphi} \|^2 .$$

Finally suppose that we have equality in Conjecture 5.2 which means that

$$(|G|_{p'})^2 = < c, \overline{\varphi} >^2 = < \overline{\varphi}, \overline{\varphi} >^2 .$$

Thus the inequality in the Cauchy-Schwarz estimation (7) is an equality. This forces

$$c = s\,\overline{\varphi}$$

for some $s \in \mathbb{N}$. If we define $\psi(g) = |G|_p$ for p'-elements $g \in G$ and $\psi(g) = 0$ otherwise then ψ is a generalized character, by ([9], Proposition 15.9). Hence ψ is a $\mathbb{Z}$-linear combination of the characters Φ_{φ_i} according to ([9], Proposition 15.13). This shows that the greatest common divisor of the degrees of Φ_{φ_i} is equal to $|G|_p$. It follows $s = 1$, hence $\Phi_{\varphi_i}(1) = |G|_p\,\varphi_i(1)$. Thus

$$\Phi_{\varphi_i} = \Phi_1\,\varphi_i$$

for $i = 1, \ldots, k$ and by a result of Brockhaus [3], the Sylow p-subgroup of G is normal which completes the proof. □

Remark 5.4 Conjecture 5.2 says that $|G|_{p'} \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2$. The equality

$$|G|_{p'} = \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1)^2$$

is equivalent to $c_\varphi = \varphi(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$. This should happen if and only if a Sylow p-subgroup is normal in G. The inequality (1) implies

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1) \leq \sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi^{\mathrm{Fix}(P)}(1)\varphi(1) \ \leq \ |G|_{p'}.$$

Thus we have

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi^{\mathrm{Fix}(P)}(1)\varphi(1) = |G|_{p'}$$

if and only if $c_\varphi = \varphi^{\mathrm{Fix}(P)}(1)$ for all $\varphi \in \mathrm{IBr}_p(G)$. According to section 3 this happens if and only if 1_P^G is completely reducible. Unfortunately we do not have a full classification in terms of the group structure in this case. Finally,

$$\sum_{\varphi \in \mathrm{IBr}_p(G)} \varphi(1) = |G|_{p'}$$

if and only if $c_\varphi = 1$ for all $\varphi \in \mathrm{IBr}_p(G)$. A full characterization of groups which satisfy this condition is given in section 4.

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