BRAUER CHARACTERS OF PRIME POWER DEGREES AND CONJUGACY CLASSES OF PRIME POWER LENGTHS

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1. Introduction and Results

Let G be a finite group and let p be a prime. It is well known that the degree of an irreducible p-Brauer character of G may fail to divide $|G|$, and so it is usually very difficult to say anything about divisibility properties of the degrees of Brauer characters. One can hardly recall anything beyond the facts about ordinary characters of defect zero, and the celebrated theorem of Michler [M1] which asserts that the Sylow p-subgroups of G are normal in G if and only if all irreducible p-Brauer characters of G have degree coprime to p. In this paper we will look at another extreme case where the degree of any irreducible p-Brauer character of G is a power of a fixed prime s. If $p = 0$, then Thompson's theorem [T] implies that G is s-nilpotent, with abelian normal s-complement. A block-theoretic version of this result has been obtained by Navarro and Robinson [NR].

Theorem 1. *Let G be a finite non-abelian simple group, $p = 0$ or a prime, and let s be any prime. Then the degree of any irreducible p-Brauer character of G is a power of s if and only if $p = s = 2$ and G is one of the following groups: $SL_2(2^f)$ with $f \geq 2$, ${}^2B_2(2^{2f+1})$ or $Sp_4(2^f)'$ with $f \geq 1$, and $PSL_2(q)$ for $q \geq 5$ a Fermat prime.*

This theorem implies the following immediate consequence:

Corollary 2. *Let G be a finite group, $p = 0$ or a prime, and let s be any prime. Assume that the degree of any irreducible p-Brauer character of G is a power of s, and that either p or s is not 2. Then G is solvable.*

Proof. Let G be a minimal counterexample and let N be a minimal normal subgroup of G. Then $N = S \times \ldots \times S$ for some non-abelian simple group S. Since any irreducible p-Brauer character of G has degree a power of s, the same is true for N and so for S as well. But this contradicts Theorem 1. $\square$

When dealing with the arithmetic of character degrees one is tempted to ask similiar questions for the conjugacy class lengths. In contrast to Theorem 1 whose proof

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depends on the classification of finite simple groups the answer can be given more easily.

Theorem 3. *Let p and s are primes. The following statements are equivalent.*

a) $|G : C_G(x)|$ *is a power of* s *for all* p'*-elements* $x \in G$.

b) G *is solvable and*

  (i) *the Hall* p'*-subgroups of* G *are abelian if* $s = p$ *or*

  (ii) $G = O_p(G) \times O_{p'}(G)$ *where the Hall* s'*-subgroups of* $O_{p'}(G)$ *are abelian if* $s \neq p$.

Proof. In both cases the implication b) to a) is obvious. Thus we may assume that condition a) is given. In order to show that G is solvable we argue by induction on $|G|$. By a result of Burnside (see [H], Kap. V, Satz 7.2), the group G is not simple since there is a class length of s-power order. Let $1 < N \lhd G$. Since the lengths of the conjugacy classes of N and G/N are divisors of the lengths of conjugacy classes of G, both N and G/N are solvable by induction. Thus G is solvable.

First suppose that $s = p$. Let H be a p-complement in G and let N be a minimal normal subgroup of G. Again by induction on $|G|$, we may assume that N is a p'-group and that N is the only minimal normal subgroup. Since G is solvable, N is a q-group for some prime $q \neq p$. Thus we have

$$N \leq O_q(G) = \mathrm{Fit}(G).$$

By the assumption in a), $C_G(x) \geq \mathrm{Fit}(G)$ for all p'-elements x. Since for solvable groups the centralizer of the Fitting group is contained in the Fitting group we get

$$\mathrm{Fit}(G) \geq C_G(\mathrm{Fit}(G)) \geq \langle x \mid x \text{ a } p'\text{-element}\rangle.$$

In particular, $\mathrm{Fit}(G)$ is a Hall p'-subgroup of G. The assumption that all p'-conjugacy classes have p-power sizes forces $\mathrm{Fit}(G)$ to be abelian.

Now let $s \neq p$. By a result of Camina [C] we have $G = O_p(G) \times O_{p'}(G)$. The assumption on the lengths of the p'-conjugacy classes yields that the Hall s'-subgroups of $O_{p'}(G)$ are abelian. □

As an immediate consequence we get that $|G : C_G(x)|$ is a suitable power of a fixed prime s for all $x \in G$ if and only if G is nilpotent with abelian Hall s'-subgroups. Furthermore there is a relation between the statements of Theorem 1 and 3.

Corollary 4. *Let p be a prime and suppose that the lengths of all p'-conjugacy classes are powers of p. Then the degrees of all irreducible p-Brauer characters are powers of p as well.*

Proof. By Theorem 3, the group G is solvable with an abelian p-complement H. Let Φ denote the projective character corresponding to an irreducible p-Brauer character φ. By a result of Fong (see [F], Chap. X, 3.2), we have

$$\Phi(1) = |G|_p \varphi(1)_{p'}$$

where $|G|_p$ denotes the p-part of the order of G and $\varphi(1)_{p'}$ the p'-part of the degree of φ. By a further result, also due to Fong (see [F], Chap. X, 3.4), Φ is induced from an irreducible so-called Fong character χ of a p-complement of G. Since H is abelian $\chi(1) = 1$, hence $\Phi(1) = |G|_p$. This shows that $\varphi(1)_{p'} = 1$. Thus all irreducible Brauer characters have p-power degrees. □

The converse of Corollary 4 fails to be true.

Example 5. Let $G = (C_3 \times C_3) : SL_2(3)$ where $SL_2(3)$ acts faithfully on the normal subgroup $C_3 \times C_3$ and let $p = 2$. One easily checks that all irreducible 2-Brauer characters of G have 2-power degrees and that there is a 3-conjugacy class whose size is not a power of 2. Clearly, the Sylow 3-subgroups of G are not abelian which is in accordance with Theorem 3.

2. Proof of Theorem 1

First we note that in the proof of Theorem 1 we may assume that p divides $|G|$. Indeed, otherwise by the Ito-Michler theorem the Sylow t-subgroups of G are normal and abelian for some prime divisor $t \neq s$ of $|G|$ and so G cannot be simple. Furthermore, if $p \neq s$ then by Michler's theorem [M1] the Sylow p-subgroups of G are normal and so G is not simple.

From now on, we will assume that $p = s$ divides $|G|$ and G is a simple group satisfying the hypothesis of Theorem 1. Let $\mathrm{IBr}_p(G)$ denote the set of all irreducible p-Brauer characters of G. We will also use the classification of finite simple groups.

Lemma 6. *G cannot be a sporadic simple group.*

Proof. Assume the contrary. The irreducible p-Brauer characters are not completely known for several sporadic groups. So instead, we will use the smallest degree d of nontrivial irreducible p-Brauer characters of G as given in [Jan]. Most of the times, this degree d is not a power of p. If it is: $d = p^a$ for some a, then we will find another $\varphi \in \mathrm{IBr}_p(G)$ such that $\varphi(1)$ is a not a p-power. Consider for example the case $G = Co_2$. Then from [Jan] we see that $d = p = 23$. The irreducible 23-Brauer characters of Co_2 are available [ModAt], and we can choose $\varphi(1) = 275$. Other cases are treated similarly. □

The next Lemma follows directly from a result of Bessenrodt [Bes].

Lemma 7. *Assume $G = \mathsf{A}_n$ is an alternating group. Then $p = s = 2$ and $n = 5, 6$.* □

Proposition 8. *Suppose the group G in Theorem 1 is a simple group of Lie type in characteristic $r \neq p$. Then $p = s = 2$ and $G = PSL_2(q)$ with $q = 9$ or q a Fermat prime.*

Proof. (i) By the main results of [M2] and [W], G has an irreducible complex character χ of p-defect zero. Since $\chi(\mathrm{mod}\, p) \in \mathrm{IBr}_p(G)$, by the hypothesis we must have $\chi(1) = p^m$ for some integer m. By Lemma 7 we may exclude all the groups G which are isomorphic to some alternating group A_n. Next we can apply the main result of [MZ] to the character χ of prime power degree and arrive at the following possibilities for $(G, \chi(1))$. These possibilities will be analyzed separately.

(ii) Case I: $(G, \chi(1))$ is one of the following: $(G_2(3), 64)$, $(SU_3(3), 32)$, $(SL_3(3), 16)$, $({}^2F_4(2)', 27)$, $(Sp_6(2), 27)$, $(Sp_6(2), 7)$. In the first three cases, $\mathrm{IBr}_2(G)$ contains characters of degree 14, respectively, 6, 12. In the fourth and the fifth cases, $\mathrm{IBr}_3(G)$ contains a character of degree 26, respectively 14. Finally, in the sixth case, $\mathrm{IBr}_7(G)$ contains a character of degree 15. In each of these cases we therefore get a contradiction.

Case II: $(G, \chi(1))$ is one of the following: $(SU_4(2), 81)$, $(PSp_4(3), 64)$, $(SL_3(2), 7)$, $(PSL_2(7), 8)$. (This case arises because of the isomorphism between simple groups of Lie type in different characteristics.) Then $\mathrm{IBr}_p(G)$ contains a character of degree 14, 14, 3, and 3, respectively, which is not a p-power.

(iii) In the remaining cases, it suffices to find a character $\psi \in \mathrm{Irr}(G)$ such that $\chi(1)|\psi(1)$ but $\psi(1) > \chi(1)$. Indeed, since $\chi(1) = |G|_p$, such ψ has p-defect zero and so $\psi(\mathrm{mod}\, p) \in \mathrm{IBr}_p(G)$. But $\psi(1) > \chi(1)$, hence $\psi(1)/\chi(1)$ cannot be a p-power. Thus $\psi(1)$ is not a p-power, contradicting the hypothesis of the theorem.

Case IIIa: $(G, \chi(1)) = (PSp_{2n}(3), (3^n - 1)/2)$ for some (prime) $n \geq 2$. Then we can choose ψ to be an irreducible constituent of degree $(3^n - 1)(3^n + 3)/4$ of the rank 3 permutation character of G, cf. [ST].

Case IIIb: $(G, \chi(1)) = (PSp_{2n}(q), (q^n + 1)/2)$ for some odd q and $n \geq 2$. Then we again choose ψ to be an irreducible constituent of degree $(q^n + 1)(q^n - q)/2(q - 1)$ of the rank 3 permutation character of G.

Case III: $(G, \chi(1)) = (PSL^{\epsilon}_n(q), (q^n - \epsilon)/(q - \epsilon))$ for some prime $n > 2$, and $\epsilon = +$ for $G = PSL_n(q)$ and $\epsilon = -$ for $G = PSU_n(q)$. If $n \geq 4$, then we can choose ψ to be the (irreducible) unipotent character labeled by the partition $(n - 2, 2)$, of degree $(q^n - \epsilon)(q^{n-1} - q^2)/(q - \epsilon)(q^2 - 1)$. Notice that $\chi(1)|\psi(1)$ since n is odd. Now assume that $n = 3$. We may assume that $(q, \epsilon) \neq (2, -)$ as $PSU_3(2)$ is solvable. Observe that if $3|(q - \epsilon)$, then $(q^3 - \epsilon)/(q - \epsilon)$ is divisible by 3 and greater than 20, and so it is not a p-power. Hence $3 \not| (q - \epsilon)$ and so $GL^{\epsilon}_3(q) \simeq G \times C_{q-\epsilon}$. It is well known that $GL^{\epsilon}_3(q)$ has an irreducible character ρ of degree $q(q^3 - \epsilon)/(q - \epsilon)$. Now we can take $\psi = \rho\downarrow_G$.

(iii) Finally, we consider the case where $S = PSL_2(q)$. Using Lemma 7 we may assume that $q \geq 7$ and $q \neq 9$.

Suppose $2|q \geq 8$. Then $\mathrm{IBr}_p(G)$ contains characters of degree $q - 1$, cf. [B], whence $q - 1 = p^a$. Since $2|q$, p is odd, and so $p \not| (q + 1)$. In this case, the Steinberg character St of G is irreducible modulo p. Clearly, $St(1) = q$ cannot be a p-power.

Now we suppose that q is odd. Again, if $p \not| (q+1)$, then the Steinberg character St of G is irreducible modulo p and $St(1) = q$ is not a p-power. So $p|(q+1)$. Assume $q \equiv 3 (\mathrm{mod}\, 4)$. Then $\mathrm{IBr}_p(G)$ contains a character of odd degree $(q-1)/2 > 1$, cf. [B] . Since $p|(q+1)$, $(q-1)/2$ cannot be a p-power.

Finally, assume that $q \equiv 1 (\mathrm{mod}\, 4)$ and $q > 9$. Again by [B], $\mathrm{IBr}_p(G)$ contains a character of even degree $(q-1)/2$ or $q-1$. It follows that $p = 2$ and $q = 2^a + 1$. Since $q > 9$ is a prime power, we conclude that q is a Fermat prime. Moreover, in this case all characters in $\mathrm{IBr}_2(G)$ have degrees 1, $q-1 = 2^a$, or $(q-1)/2 = 2^{a-1}$. □

Proposition 9. *Suppose the group G in Theorem 1 is a simple group of Lie type in characteristic p. Then $p = s = 2$ and $G \in \{SL_2(2^f), {}^2B_2(2^f), Sp_4(2^f)'\}$.*

Proof. View G as $L/Z(L)$, where L is the Lie-type group of simply connected type corresponding to G. Also, let ${}^{(p)}$ denote the Frobenius twist induced by the map $x \mapsto x^p$.

(i) First we consider the case $G = PSL_n^\epsilon(q)$, with $\epsilon = +$ for PSL and $\epsilon = -$ for PSU, and $n \geq 2$. Then the largest irreducible subquotient A of the adjoint module for $L = SL_n^\epsilon(q)$ has dimension $n^2 - 2$ if $p|n$ and $n^2 - 1$ if $p \not| n$, cf. [MMT, Table 1.1]. Furthermore, $Z(L)$ acts trivially on A, so we can view A as an irreducible G-module.

Assume we are in the former case. By assumption, $p|(n^2 - 2)$, hence $p = 2$ and $n = 2k$ for some integer k. Now $\dim(A) = 4k^2 - 2 \equiv 2(\bmod\, 4)$ is a 2-power, which is possible only when $k = 1$, i.e. $G \simeq SL_2(2^f)$.

Assume now that we are in the latter case. Then $\dim(A) = (n-1)(n+1)$ is a p-power, which is possible only when $(n, p) = (2, 3)$ or $(3, 2)$. Consider the first possibility: $(n, p) = (2, 3)$. Since G is simple, $q = 3^f \geq 9$. Let $\mathcal{N} := \overline{\mathbb{F}}_q^n$ denote the natural module for L. By the Steinberg tensor product theorem, $\mathcal{N} \otimes \mathcal{N}^{(3)}$ is an irreducible L-module, of dimension 4 and with trivial action of $Z(L)$. Thus $\mathrm{IBr}_p(G)$ contains a character of degree 4, a contradiction. Consider the second possibility: $(n, p) = (3, 2)$. Since G is simple, $(q, \epsilon) \neq (2, -)$. If $3 \not| (q - \epsilon)$, then $Z(L) = 1$ and so $\mathcal{N}$ is an irreducible G-module of dimension 3, not a 2-power. Thus $3|(q - \epsilon)$; in particular, $q = 2^f \geq 4$. By the Steinberg tensor product theorem, $\mathcal{N} \otimes \mathcal{N}^{(2)}$ is an irreducible L-module, of dimension 9 and with trivial action of $Z(L)$. Thus $\mathrm{IBr}_p(G)$ contains a character of degree 9, a contradiction.

(ii) Suppose $G = PSp_{2n}(q)'$ with $n \geq 2$. Then the largest irreducible subquotient A of the adjoint module for $L = Sp_{2n}(q)$ has dimension $n(2n+1)$ if $p \neq 2$, $(n-1)(2n+1)$ if $p = 2 \not| n$, and $n(2n-1) - 2$, if $p = 2|n$, cf. [MMT, Table 1.1]. Again, $Z(L)$ acts trivially on A, so we can view A as an irreducible G-module. Clearly, neither $n(2n+1)$ nor $(n-1)(2n+1)$ can be a p-power. Hence $p = 2|n$. The case $n = 2$ is one of the conclusions of the proposition, so we assume that $n \geq 4$. Since $p = 2$, $Z(L) = 1$, so the natural module $\mathcal{N} = \overline{\mathbb{F}}_q^{2n}$ is also a G-module. It follows that $2n = \dim(\mathcal{N})$ is a 2-power, i.e. $n = 2^a$ with $a \geq 2$. In this case, $26 \leq \dim(A) = 2^{2a+1} - 2^a - 2 \equiv 2(\bmod 4)$ and so $\dim(A)$ cannot be a 2-power.

(iii) Suppose $G = P\Omega_n^{\pm}(q)$ with $n \geq 7$, or ${}^3D_4(q)$ (in which case we set $n = 8$); furthermore, $2|n$ if $p = 2$. Then the largest irreducible subquotient A of the adjoint module for $L = Spin_n^{\pm}(q)$, respectively ${}^3D_4(q)$, has dimension $n(n-1)/2$ if $p \neq 2$, $(n-2)(n+1)/2$ if $p = 2 \not| (n/2)$, and $n(n-1)/2 - 2$, if $p = 2|(n/2)$, cf. [MMT, Table 1.1]. Again, $Z(L)$ acts trivially on A, so we can view A as an irreducible G-module. Clearly, neither $n(n-1)/2$ nor $(n-2)(n+1)/2$ can be a p-power. Hence $p = 2|n$. Now we can argue as in (iii) to arrive at a contradiction.

(iv) The case $G = {}^2B_2(q)$ is listed in the conclusions of the proposition. Now we consider the (remaining) exceptional groups of Lie type: $G_2(q)$, respectively, ${}^2G_2(q)$ with $q \geq 27$, F_4, ${}^2F_4(q)'$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$. According to [KL, Table 5.4.B], G has an irreducible module of dimension $14 - 7\delta_{p,3}$, 7, $26 - \delta_{p,3}$, 26, $78 - \delta_{p,3}$, $78 - \delta_{p,3}$, $133 - \delta_{p,2}$, 248, respectively (where $\delta_{p,t}$ is the Kronecker symbol). None of these dimensions can be a p-power. □

Proof of Theorem 1. The "only if" part of the theorem is an immediate consequence of Lemmas 6, 7, and Propositions 8, 9. The "if" part can be verified directly. □

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